

SPLIT PRINCIPLES, LARGE CARDINALS, SPLITTING FAMILIES AND SPLIT IDEALS

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ABSTRACT. We introduce split principles and show that their negations provide simple combinatorial characterizations of large cardinal properties. As examples, we show how inaccessibility, weak compactness, subtlety, almost ineffability and ineffability can be characterized. Two-cardinal versions of these principles allow us to characterize when the two-cardinal versions of these properties hold: when κ is almost λ -ineffable, λ -ineffable, and we can also characterize mild ineffability by a slightly modified version of the principle. We also characterize when κ is λ -Shelah. Using these facts, we get characterizations of strong compactness and supercompactness. Many of these characterizations can be expressed by saying that certain infinitary splitting numbers are large. The split principles naturally give rise to some large cardinal properties which, in the two-cardinal context, seem to be new. These split principles come with certain ideals, and one of the split principles characterizing a version of κ being λ -Shelah gives rise to a normal ideal on $\mathcal{P}_\kappa\lambda$. We also explore relationships between these large cardinal concepts and partition relations.

1. INTRODUCTION

A version of the split principle was first considered by Fuchs, Gitman, and Hamkins in the course of their work on [FGH14], the intended use being the construction of ultrafilters with certain properties. It turned out that those constructions were not needed, and later, the first author observed that the split principle is an “anti-large-cardinal axiom” which characterizes the failure of a regular cardinal to be weakly compact. In the present paper, we consider several versions of the principle that provide simple combinatorial characterizations of the failure of various large cardinal properties.

The split principles for κ say that there is a sequence $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ that splits every subset A of κ that’s in some sense large into two subsets of A that are also large in some sense, meaning that there is one ordinal β such that for many $\alpha \in A$, β belongs to d_α , but also, for many $\alpha \in A$, β belongs to $\alpha \setminus d_\alpha$. By varying the meanings of “large” in this principle, we obtain a host of natural split principles.

In section 2, we introduce the original split principle in detail, and we show that if the notions of largeness used are reasonable (namely, the largeness of the sets split into is determined by a tail of the sets), then the failure of the split principle at κ says that the corresponding splitting number is larger than κ . The notion of splitting number and splitting family here is the obvious generalization to arbitrary κ of the well-known concepts at ω . We then show that the nonexistence of a κ -list that splits unbounded subsets of a regular cardinal κ into unbounded sets is equivalent to the weak compactness of κ . This is also equivalent to saying that the corresponding splitting number is greater than κ . We then show that the nonexistence of a sequence that splits stationary subsets of a regular cardinal into various classes (anything between the class of nonempty sets and the class of

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stationary subsets) characterizes its ineffability, and a regular cardinal κ is almost ineffable iff there is no sequence that splits unbounded subsets into nonempty subsets. This latter characterization does not correspond to a statement about splitting numbers in any obvious way. We also obtain a characterization of subtle cardinals.

In Section 3, we consider split principles asserting the existence of lists that split subsets of $\mathcal{P}_\kappa\lambda$. The nonexistence of a sequence that splits unbounded subsets of $\mathcal{P}_\kappa\lambda$ into unbounded is a large cardinal concept which we call wild ineffability and which is situated somewhere between mild and almost ineffability (and we don't know where it lies - it may be equivalent to one of these). We characterize wild ineffability in terms of delayed coherence properties of $\mathcal{P}_\kappa\lambda$ -lists, showing that the concept is a natural one, and we show that it is implied by the partition property $\text{Part}(\kappa, \lambda)_{<}^3$. We give split principle characterizations of the two cardinal versions of almost ineffability and ineffability as before. Versions of the split principle which postulate that functions $F : \lambda \rightarrow \lambda$ can be split allow us to characterize versions of κ being λ -Shelah. We call the one corresponding to the failure of the functional split principle splitting unbounded sets into unbounded sets wild Shelahness.

Finally, in Section 4, we introduce the $\mathcal{P}_\kappa\lambda$ -ideals of sets on which the split principles hold (and thus, where the corresponding large cardinal properties fail), and show that the ideal associated to the functional split principle is strongly normal. A lot of our analysis of $\mathcal{P}_\kappa\lambda$ split principles and the ideals they give rise to is closely related to prior work of Donna Carr ([Car81, Car87]) as well as DiPrisco and Zwicker ([DZ80]).

2. SPLITTING SUBSETS OF κ

Let κ be a cardinal. We shall use the terminology of [Wei10], and refer to a sequence of the form $\langle d_\alpha \mid \alpha < \kappa \rangle$ as a κ -list if for all $\alpha < \kappa$, $d_\alpha \subseteq \alpha$.

Definition 2.1. Let \mathcal{A} and \mathcal{B} be families of subsets of κ . The principle $\text{Split}_\kappa(\mathcal{A}, \mathcal{B})$ says that there is a κ -list \vec{d} that *splits \mathcal{A} into \mathcal{B}* , meaning that for every $A \in \mathcal{A}$, there is a $\beta < \kappa$ such that both $A_{\beta, \vec{d}}^+ = A_\beta^+ = \{\alpha \in A \mid \beta \in d_\alpha\}$ and $A_{\beta, \vec{d}}^- = A_\beta^- = \{\alpha \in A \mid \beta \in \alpha \setminus d_\alpha\}$ are in \mathcal{B} . In this case, we also say that β splits A into \mathcal{B} with respect to \vec{d} .

We abbreviate $\text{Split}_\kappa(\mathcal{A}, \mathcal{A})$ by $\text{Split}_\kappa(\mathcal{A})$.

The collections of all unbounded, all stationary and all κ -sized subsets of κ are denoted by unbounded , stationary and $[\kappa]^\kappa$ respectively.

The idea is that \mathcal{A} and \mathcal{B} are collections of “large” sets, and $\text{Split}_\kappa(\mathcal{A}, \mathcal{B})$ says that there is one κ -list that can split any set that's large in the sense of \mathcal{A} into two disjoint sets that are large in the sense of \mathcal{B} , in the uniform way described in the definition. There is a very close connection to the concept of splitting families, which can be made explicit after considering a wider range of split principles, in which we drop the assumption that the sequence is a κ -list.

Definition 2.2. Let κ and τ be cardinals, and let \mathcal{A} and \mathcal{B} be families of subsets of κ . The principle $\text{Split}_{\kappa, \tau}(\mathcal{A}, \mathcal{B})$ says: there is a sequence $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ of subsets of τ such that for every $A \in \mathcal{A}$, there is an ordinal $\beta < \tau$ such that the sets $\tilde{A}_\beta^+ = \tilde{A}_{\beta, \vec{d}}^+ = \{\alpha \in A \mid \beta \in d_\alpha\}$ and $\tilde{A}_\beta^- = \tilde{A}_{\beta, \vec{d}}^- = \{\alpha \in A \mid \beta \notin d_\alpha\}$ belong to \mathcal{B} . Such a sequence is called a $\text{Split}_{\kappa, \tau}(\mathcal{A}, \mathcal{B})$ sequence.

If $\mathcal{A} = \mathcal{B}$, we don't mention \mathcal{B} in the notation. Thus, $\text{Split}_{\kappa, \tau}(\mathcal{A}, \mathcal{A})$ is $\text{Split}_{\kappa, \tau}(\mathcal{A})$.

In this context, a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ is an $(\mathcal{A}, \mathcal{B})$ -splitting family for κ if for every $A \in \mathcal{A}$, there is an $S \in \mathcal{F}$ such that both $A \cap S$ and $A \setminus S$ belong to \mathcal{B} . The $(\mathcal{A}, \mathcal{B})$ -splitting number, denoted

$\mathfrak{s}_{\mathcal{A},\mathcal{B}}(\kappa)$ is the least size of an $(\mathcal{A},\mathcal{B})$ -splitting family. We write $\mathfrak{s}_{\mathcal{A}}(\kappa)$ for $\mathfrak{s}_{\mathcal{A},\mathcal{A}}(\kappa)$. $\mathfrak{s}(\kappa)$ stands for $\mathfrak{s}_{[\kappa]^\kappa}(\kappa)$.

Of course, $\mathfrak{s} = \mathfrak{s}(\omega)$ is a well-known cardinal characteristic of the continuum. Several authors have considered the generalization $\mathfrak{s}(\kappa)$, for uncountable κ , see [Suz93], [Zap97]. Note that for regular κ , $\text{unbounded} = [\kappa]^\kappa$. We'll only use $[\kappa]^\kappa$ when κ is singular.

We will first explore the relation between the two types of split principles introduced so far.

Observation 2.3. *Let κ be a cardinal, and let \mathcal{A} and \mathcal{B} be collections of subsets of κ such that for all $B \subseteq \kappa$ and all $\beta < \kappa$, $B \in \mathcal{B}$ iff $B \setminus \beta \in \mathcal{B}$ (" \mathcal{B} is independent of initial segments"). Then $\text{Split}_{\kappa,\kappa}(\mathcal{A},\mathcal{B})$ is equivalent to $\text{Split}_\kappa(\mathcal{A},\mathcal{B})$.*

Proof. If \vec{e} is a $\text{Split}_\kappa(\mathcal{A},\mathcal{B})$ -sequence, clearly it is also a $\text{Split}_{\kappa,\kappa}(\mathcal{A},\mathcal{B})$ -sequence.

For the other direction, let \vec{d} be a $\text{Split}_{\kappa,\kappa}(\mathcal{A},\mathcal{B})$ sequence. Define the κ -list \vec{e} by $e_\alpha = d_\alpha \cap \alpha$. It follows that \vec{e} is a $\text{Split}_\kappa(\mathcal{A},\mathcal{B})$ sequence, because if $A \in \mathcal{A}$ and $\beta < \kappa$ is such that $\tilde{A}_{\beta,\vec{d}}^+$ and $\tilde{A}_{\beta,\vec{d}}^-$ are in \mathcal{B} by $\text{Split}_{\kappa,\kappa}(\mathcal{A},\mathcal{B})$, then $A_{\beta,\vec{e}}^+ = \tilde{A}_{\beta,\vec{d}}^+ \setminus (\beta+1)$, and similarly, $A_{\beta,\vec{e}}^- = \tilde{A}_{\beta,\vec{d}}^- \setminus (\beta+1)$. It follows from our assumption on \mathcal{B} that $A_{\beta,\vec{e}}^+$ and $A_{\beta,\vec{e}}^-$ are in \mathcal{B} . \square

Note that unbounded , stationary and $[\kappa]^\kappa$ all are independent of initial segments. The following lemma says that the split principles can be viewed as statements about the sizes of the corresponding splitting numbers.

Lemma 2.4. *Let κ and τ be cardinals, and let \mathcal{A}, \mathcal{B} be families of subsets of κ . Then $\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$ holds iff $\mathfrak{s}_{\mathcal{A},\mathcal{B}}(\kappa) \leq \tau$.*

Note: In other words, $\mathfrak{s}_{\mathcal{A},\mathcal{B}}(\kappa)$ is the least τ such that $\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$ holds.

Proof. For the direction from right to left, if $\mathcal{S} = \{x_\alpha \mid \alpha < \tau\}$ is an $(\mathcal{A},\mathcal{B})$ -splitting family for κ , then we can define a sequence $\langle d_\alpha \mid \alpha < \tau \rangle$ of subsets of τ by setting

$$d_\alpha = \{\gamma < \tau \mid \alpha \in x_\gamma\}.$$

Then \vec{d} is a $\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$ sequence, because if $A \in \mathcal{A}$, then there is a $\beta < \tau$ such that both $A \cap x_\beta$ and $A \setminus x_\beta$ belong to \mathcal{B} , but $A \cap x_\beta = \tilde{A}_{\beta,\vec{d}}^+$ and $A \setminus x_\beta = \tilde{A}_{\beta,\vec{d}}^-$ so we are done.

Conversely, if \vec{d} is a $\text{Split}_{\kappa,\tau}(\mathcal{A},\mathcal{B})$ sequence, then for each $\gamma < \tau$, we define a subset x_γ of κ by

$$x_\gamma = \{\alpha < \kappa \mid \gamma \in d_\alpha\}.$$

Then $\mathcal{S} = \{x_\gamma \mid \gamma < \tau\}$ is an $(\mathcal{A},\mathcal{B})$ -splitting family for κ , because if $A \in \mathcal{A}$, then there is a $\beta < \tau$ such that both $\tilde{A}_{\beta,\vec{d}}^+$ and $\tilde{A}_{\beta,\vec{d}}^-$ are in \mathcal{B} , but as before, $\tilde{A}_{\beta,\vec{d}}^+ = A \cap x_\beta$ and $\tilde{A}_{\beta,\vec{d}}^- = A \setminus x_\beta$. \square

What this proof shows is that if $X \subseteq \kappa \times \lambda$ is a set and we let, for $\beta < \lambda$, be $X^\beta = \{\alpha < \kappa \mid \langle \alpha, \beta \rangle \in X\}$ be the horizontal section at height β , and for $\alpha < \kappa$, $X_\alpha = \{\beta < \lambda \mid \langle \alpha, \beta \rangle \in X\}$ be the vertical section at α , then $\langle X_\alpha \mid \alpha < \kappa \rangle$ is a $\text{Split}_{\kappa,\lambda}(\mathcal{A},\mathcal{B})$ sequence iff $\langle X^\beta \mid \beta < \lambda \rangle$ is an $(\mathcal{A},\mathcal{B})$ -splitting family.

Corollary 2.5. *If \mathcal{B} is independent of initial segments, then $\text{Split}_\kappa(\mathcal{A},\mathcal{B})$ holds iff $\mathfrak{s}_{\mathcal{A},\mathcal{B}}(\kappa) \leq \kappa$.*

Observation 2.6. *Let κ be a cardinal, and let $\mathcal{A}, \mathcal{A}', \mathcal{B}, \mathcal{B}'$ be collections of subsets of κ . If $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B}' \subseteq \mathcal{B}$, then $\mathfrak{s}_{\mathcal{A},\mathcal{B}}(\kappa) \leq \mathfrak{s}_{\mathcal{A}',\mathcal{B}'}(\kappa)$.*

Proof. Under the assumptions stated, every $(\mathcal{A}',\mathcal{B}')$ -splitting family is also $(\mathcal{A},\mathcal{B})$ -splitting. \square

We are now ready to characterize inaccessible cardinals by split principles. The equivalence $1 \iff 5$ follows from work of Motoyoshi.

Lemma 2.7. *Let κ be an uncountable regular cardinal. The following are equivalent:*

- (1) κ is inaccessible.
- (2) $\text{Split}_{\kappa, \tau}(\text{stationary}, \text{nonempty})$ fails for every $\tau < \kappa$. Equivalently, $\mathfrak{s}_{\text{stationary}, \text{nonempty}}(\kappa) \geq \kappa$.
- (3) $\text{Split}_{\kappa, \tau}(\text{stationary})$ fails for every $\tau < \kappa$. Equivalently, $\mathfrak{s}_{\text{stationary}}(\kappa) \geq \kappa$.
- (4) $\text{Split}_{\kappa, \tau}(\text{unbounded}, \text{nonempty})$ fails for every $\tau < \kappa$. Equivalently, $\mathfrak{s}_{\text{unbounded}, \text{nonempty}}(\kappa) \geq \kappa$.
- (5) $\text{Split}_{\kappa, \tau}(\text{unbounded})$ fails for every $\tau < \kappa$. Equivalently, $\mathfrak{s}_{\text{unbounded}}(\kappa) \geq \kappa$.

It follows that for any collection \mathcal{B} with $\text{stationary} \subseteq \mathcal{B} \subseteq \text{nonempty}$, these conditions are equivalent to the failure of $\text{Split}_{\kappa, \tau}(\text{stationary}, \mathcal{B})$, and similarly for any \mathcal{B} with $\text{unbounded} \subseteq \mathcal{B} \subseteq \text{nonempty}$, they are equivalent to the failure of $\text{Split}_{\kappa, \tau}(\text{unbounded}, \mathcal{B})$.

Moreover, 2., 4. are equivalent to κ being inaccessible even if κ is not assumed to be regular (not even to be a cardinal).

Proof. The equivalence $1. \iff 5.$ follows from previously known results as follows. According to [Suz93], it was shown in [Mot92] that for an uncountable regular cardinal κ , κ is inaccessible iff $\mathfrak{s}(\kappa) \geq \kappa$ (see [Zap97, Lemma 3] for a proof). By Lemma 2.4, this is equivalent to saying that for no $\tau < \kappa$ does $\text{Split}_{\kappa, \tau}(\text{unbounded})$ hold.

However, we will give a self-contained proof here.

The implications $2. \implies 3.$ and $2. \implies 4. \implies 5.$ are trivial.

To prove $1. \implies 2.$, let κ be inaccessible, and let $\tau < \kappa$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \tau \rangle$ be a sequence of subsets of τ . We will show that \vec{d} does not split stationary subsets of κ into nonempty sets. Since $2^\tau < \kappa$, it follows that there is a stationary set $S \subseteq \kappa$ and a set $e \subseteq \tau$ such that for all $\alpha \in S$, $d_\alpha = e$. Let $\beta < \tau$. Then $\tilde{S}_\beta^+ = \{\alpha \in S \mid \beta \in e\}$ and $\tilde{S}_\beta^- = \{\alpha \in S \mid \beta \notin e\}$, so one of these is S and the other is \emptyset . So β does not split S into nonempty sets.

To show $5. \implies 1.$, suppose κ is not inaccessible. Since κ is assumed to be regular, it follows that there is a $\tau < \kappa$ such that $2^\tau \geq \kappa$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \tau \rangle$ be a sequence of distinct subsets of λ . By 5, \vec{d} is not a $\text{Split}_{\kappa, \tau}(\text{unbounded})$ sequence. Hence, there is an unbounded set $U \subseteq \kappa$ such that no $\beta < \tau$ splits U into unbounded sets. So, for every $\beta < \tau$, exactly one of U_β^+ and U_β^- is bounded. Let $\xi_\beta < \kappa$ be such that the bounded one is contained in ξ_β . Let $\xi = \sup_{\beta < \tau} \xi_\beta$. Then $\xi < \kappa$, since κ is regular, and we claim that for $\xi < \gamma < \delta$ with $\xi, \delta \in U$, it follows that $d_\gamma = d_\delta$. For if $\beta < \tau$ and U_β^- is bounded, then, since $\gamma \in U \setminus \xi_\beta$, it follows that $\gamma \in U_\beta^+$, which means that $\beta \in d_\gamma$, and for the same reason, $\beta \in d_\delta$. And if U_β^+ is bounded, then it follows that $\beta \notin d_\gamma$ and $\beta \notin d_\delta$. So $d_\gamma = d_\delta$, a contradiction.

A similar argument shows the final implication, $3. \implies 1.$ Assume κ were not inaccessible. Let $\tau < \kappa$ be such that $2^\tau \geq \kappa$. Let $\vec{d} = \langle d_\alpha \mid \alpha < \tau \rangle$ be a sequence of distinct subsets of τ . By 3, \vec{d} is not a $\text{Split}_{\kappa, \tau}(\text{stationary})$ sequence. Hence, there is a stationary set $S \subseteq \kappa$ such that no $\beta < \tau$ splits S into stationary sets. So, for every $\beta < \tau$, exactly one of S_β^+ and S_β^- is nonstationary. Let C_β be a club in κ , disjoint from the nonstationary one of the two. Let $T = S \cap \bigcap_{\beta < \tau} C_\beta$. This is a stationary set, and we claim that for $\gamma < \delta$ are both in T , it follows that $d_\gamma = d_\delta$. For if $\beta < \tau$ and S_β^- is nonstationary, then, since $\gamma \in S \cap C_\beta$, it follows that $\gamma \in S_\beta^+$, which means that $\beta \in d_\gamma$, and for the same reason, $\beta \in d_\delta$. And if S_β^+ is nonstationary, then it follows that $\beta \notin d_\gamma$ and $\beta \notin d_\delta$. So $d_\gamma = d_\delta$, a contradiction.

The claim about families \mathcal{B} with $\text{stationary} \subseteq \mathcal{B} \subseteq \text{nonempty}$ follows from the equivalence of 2. and 3, and the claim about \mathcal{B} with $\text{unbounded} \subseteq \mathcal{B} \subseteq \text{nonempty}$ follows from the equivalence of 4. and 5.

For the last claim, it suffices to show that 4. implies that κ is regular. But this is obvious, since if $\langle \xi_\alpha \mid \alpha < \text{cf}(\kappa) \rangle$ is cofinal in κ , then $\{\xi_\alpha \mid \alpha < \text{cf}(\kappa)\}$ (viewed as a collection of subsets of κ) is an $(\text{unbounded}, \text{nonempty})$ -splitting family. So $\mathfrak{s}_{\text{unbounded}, \text{nonempty}}(\kappa) \leq \text{cf}(\kappa)$, so $\text{Split}_{\kappa, \text{cf}(\kappa)}(\text{unbounded}, \text{nonempty})$ holds. So it has to be the case that $\text{cf}(\kappa) = \kappa$. \square

Recall that a regular cardinal κ is weakly compact if κ is inaccessible and the tree property $\text{TP}(\kappa)$ holds at κ , which states that every κ -tree has a cofinal branch, where a κ -tree is a tree of height κ all of whose levels have size less than κ . We will show that a regular cardinal κ is weakly compact if and only if $\text{Split}_\kappa(\text{unbounded})$ fails, for example. Toward this end, we will first define the analogue of the tree property for κ -lists.

Definition 2.8. A κ -list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ has a *cofinal branch*, or has a κ -branch, so long as there is a $b \subseteq \kappa$ such that for all $\gamma < \kappa$ there is an $\alpha \geq \gamma$ such that $d_\alpha \cap \gamma = b \cap \gamma$.

We say that the branch property $\text{BP}(\kappa)$ holds if every κ -list has a cofinal branch.

Given a κ -list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$, for each $\alpha < \kappa$ let $d_\alpha^c : \alpha \rightarrow 2$ denote the characteristic function of d_α . The sequential tree corresponding to the κ -list is given by $T_{\vec{d}} = \{d_\alpha^c \restriction \beta \mid \beta \leq \alpha < \kappa\}$, and the tree ordering is set inclusion. As is customary, we refer to a function $b : \kappa \rightarrow 2$ as a (cofinal) branch through $T_{\vec{d}}$ if for all $\gamma < \kappa$, $b \restriction \gamma \in T_{\vec{d}}$, which means that for every $\gamma < \kappa$, there is an $\alpha \geq \gamma$ such that $b \restriction \gamma = d_\alpha^c \restriction \gamma$.

Observation 2.9. A κ -list \vec{d} has a cofinal branch if and only if the corresponding tree $T_{\vec{d}}$ has a cofinal branch.

It turns out that for regular κ , the properties of a κ -list of splitting unbounded sets and having a cofinal branch are complementary.

Theorem 2.10. Let κ be regular, and let \vec{d} be a κ -list. The following are equivalent:

- (1) \vec{d} is a $\text{Split}_\kappa(\text{unbounded})$ sequence.
- (2) \vec{d} does not have a cofinal branch.

Proof. 1. \implies 2.: Assume towards a contradiction that \vec{d} has a cofinal branch $b \subseteq \kappa$, and that \vec{d} splits unbounded sets. We will first define a function $f : \kappa \rightarrow \kappa$ as follows:

$$f(\gamma) = \text{the least } \alpha \geq \gamma \text{ such that } b \cap \gamma = d_\alpha \cap \gamma.$$

Note that f is weakly increasing, thus letting $A = f''\kappa$, A is unbounded in κ . So there is $\beta < \kappa$ which splits A (with respect to \vec{d}), i.e., both of the sets $A_\beta^+ = \{f(\gamma) \in A \mid \beta \in d_{f(\gamma)}\}$ and $A_\beta^- = \{f(\gamma) \in A \mid \beta \in f(\gamma) \setminus d_{f(\gamma)}\}$ are unbounded in κ . There are two cases.

Case 1: $\beta \notin b$. Since A_β^+ is unbounded, we may choose $f(\gamma) \in A_\beta^+$ satisfying $f(\gamma) > f(\beta)$. By the weak monotonicity of f , it follows that $\gamma > \beta$. Then $\beta \in d_{f(\gamma)}$ by the definition of A_β^+ and it follows that $\beta \in d_{f(\gamma)} \cap \gamma = b \cap \gamma$, contradicting the assumption that $\beta \notin b$.

Case 2: $\beta \in b$. We may run a similar argument to the previous case in order to obtain a contradiction. In this case, we use that A_β^- is unbounded to choose $f(\gamma) \in A_\beta^-$ satisfying $f(\gamma) > f(\beta)$, so that $\gamma > \beta$, and get the contradiction that $\beta \notin d_{f(\gamma)}$ while $\beta \in b \cap \gamma = d_{f(\gamma)} \cap \gamma$.

2. \implies 1.: Assume toward a contradiction that \vec{d} does not split unbounded sets. We will get a contradiction by showing that then \vec{d} has a cofinal branch. Let $A \subseteq \kappa$ be unbounded such that no

$\beta < \kappa$ splits A (with respect to \vec{d}). Thus, for each $\beta < \kappa$, exactly one of A_β^+ or A_β^- is bounded in κ , since $A = A_\beta^+ \cup A_\beta^-$ (so since A is unbounded, it can't be that both A_β^+ and A_β^- are bounded). Now we may define our branch $b \subseteq \kappa$ as follows:

$$\beta \in b \iff A_\beta^- \text{ is bounded in } \kappa \text{ (} \iff A_\beta^+ \text{ is unbounded in } \kappa \text{.)}$$

To see that this works, note that for each $\beta < \kappa$ there is a least $\xi_\beta < \kappa$ such that either:

$$\beta \in d_\delta \text{ for all } \delta \in A \setminus \xi_\beta \text{ or } \beta \notin d_\delta \text{ for all } \delta \in A \setminus \xi_\beta,$$

The first scenario is the situation where $\beta \in b$, and the second scenario is where $\beta \notin b$.

Letting $\gamma < \kappa$ be arbitrary, using the fact that κ is regular, there is an $\alpha \in A$ such that $\alpha > \sup_{\beta < \gamma} \xi_\beta$. It follows that $b \cap \gamma = d_\alpha \cap \gamma$. To see this, let $\beta < \gamma$. We have two cases.

Case 1: $\beta \in b$. Then A_β^- is bounded, so for all $\delta \in A \setminus \xi_\beta$, $\beta \in d_\delta$, so $\beta \in d_\alpha$, since $\alpha \in A \setminus \xi_\beta$.

Case 2: $\beta \notin b$. Then A_β^+ is bounded, so $\beta \notin d_\alpha$, since $\alpha \in A \setminus \xi_\beta$.

So we have reached the desired contradiction that b is a cofinal branch. \square

Corollary 2.11. *Let κ be regular. Then κ -Split holds iff there is a sequential tree $T \subseteq {}^{<\kappa}2$ of height κ that has no cofinal branch.*

Note that a sequential tree $T \subseteq {}^{<\kappa}2$ of height κ without a cofinal branch is not necessarily an Aronszajn tree, as it doesn't even have to be a κ -tree – T may have levels of size κ . But if κ is inaccessible, then such a T is Aronszajn.

Proof. For the direction from right to left, let T be a sequential tree $T \subseteq {}^{<\kappa}2$ of height κ that has no cofinal branch. For each $\alpha < \kappa$, let s_α be a node at the α -th level of T , i.e., a sequence $s_\alpha : \alpha \rightarrow 2$ with $s_\alpha \in T$. Let d_α be the sequence s_α , viewed as a subset of α , i.e., $d_\alpha = \{\gamma < \alpha \mid s_\alpha(\gamma) = 1\}$. In other words, $s_\alpha = d_\alpha^c$. Then $T_{\vec{d}} \subseteq T$, and so, since T does not have a cofinal branch, $T_{\vec{d}}$ has no cofinal branch, which means, by Observation 2.9, that \vec{d} has no cofinal branch, and this is equivalent to saying that \vec{d} splits unbounded sets, by Theorem 2.10. So κ -Split holds.

For the converse, let \vec{d} be a κ -Split-sequence. By Theorem 2.10, \vec{d} has no cofinal branch. So by Observation 2.9, $T_{\vec{d}}$ does not have a cofinal branch, so $T_{\vec{d}}$ is as wished. \square

2.1. Weakly compact cardinals. We shall use the previous result to show that the split principle can be used to characterize weakly compact cardinals.

Corollary 2.12. *Let κ be a regular cardinal. Then $\text{Split}_\kappa(\text{unbounded}) \iff \kappa$ is not weakly compact. (This is true for $\kappa = \omega$ as well, if we consider ω to be weakly compact, which is not standard.)*

Proof. We shall show both directions of the equivalence separately.

\implies : Assuming $\text{Split}_\kappa(\text{unbounded})$, we have to show that κ is not weakly compact. If κ is not inaccessible, then we are done, so let's assume it is. By Corollary 2.11, there is a sequential tree $T \subseteq {}^{<\kappa}2$ of height κ with no cofinal branch. Since κ is inaccessible, T is a κ -tree, and thus, κ does not have the tree property, so κ is not weakly compact.

\impliedby : Let κ fail to be weakly compact. We split into two cases.

Case 1: κ is not inaccessible. Then by 2.7, there is a $\tau < \kappa$ such that $\text{Split}_{\kappa,\tau}(\text{unbounded})$ holds. This clearly implies that $\text{Split}_{\kappa,\kappa}(\text{unbounded})$ holds, and, since unbounded is independent of initial segments, this is equivalent to $\text{Split}_\kappa(\text{unbounded})$.

Case 2: κ is inaccessible but not weakly compact. Then $\text{TP}(\kappa)$ fails, and this is witnessed by a sequential tree T on ${}^{<\kappa}2$ that has no cofinal branch. Thus, κ -Split holds, by Corollary 2.11. \square

So a regular cardinal κ is weakly compact iff $\text{Split}_\kappa(\text{unbounded})$ fails. Since this is equivalent to saying that $\text{Split}_{\kappa,\kappa}(\text{unbounded})$ fails, this can be equivalently expressed by saying that $\mathfrak{s}(\kappa) > \kappa$, by Lemma 2.4. This latter characterization of weak compactness was shown in [Suz93].

It should be noted, however, before moving on to larger large cardinals, that what we really showed with our proof of Theorem 2.10 is that κ -Split is equivalent to the failure of something seemingly stronger than every κ -list having a cofinal branch, namely the failure of the *strong* branch property, which we define below.

Definition 2.13. Let $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ be a κ -list. A cofinal branch $b \subseteq \kappa$ is a *strong branch* of \vec{d} so long as there is an unbounded $U \subseteq \kappa$ such that for each $\gamma < \kappa$, there is $\alpha > \gamma$ such that for all $\delta \in U$ with $\delta > \alpha$ we have that $d_\delta \cap \gamma = b \cap \gamma$. In this case we say that the unbounded set U *guides* the cofinal branch b .

If every κ -list has a strong branch, we say $\text{SBP}(\kappa)$ holds.

Indeed the argument in the beginning of the proof of Theorem 2.10 shows that if a κ -list has a cofinal branch, then that branch is a strong branch (and it is not necessary to assume that κ is regular here). The unbounded set verifying strongness is the range of the function f in that proof. Trivially, every strong branch is a branch, and so, these concepts are equivalent for κ -lists. The situation will turn out to be less clear when dealing with $\mathcal{P}_\kappa\lambda$ -lists, as we will do in Section 3.

2.2. Ineffable cardinals.

Definition 2.14. Let \mathcal{A} be a family of subsets of κ . A κ -list $\vec{d} = \langle d_\alpha \mid \alpha < \kappa \rangle$ has an \mathcal{A} branch iff there is a set $b \subseteq \kappa$ and a set $A \in \mathcal{A}$ such that for all $\alpha \in A$, we have that $d_\alpha = b \cap \alpha$. Keeping with tradition, a stationary branch is called an *ineffable* branch, and an unbounded branch is called an *almost ineffable* branch. κ is \mathcal{A} -ineffable if κ is regular and every κ -list has an \mathcal{A} branch. stationary-ineffable cardinals are just called ineffable cardinals, and unbounded-ineffable ones are called almost ineffable.

We say that κ has the ineffable branch property, or $\text{IBP}(\kappa)$ holds, if every κ -list has an ineffable branch.

We have the following string of implications: $\text{IBP}(\kappa) \implies \text{SBP}(\kappa) \implies \text{BP}(\kappa)$. In particular, ineffable cardinals are weakly compact. We start by giving a general and uniform characterization of \mathcal{A} -ineffability.

Theorem 2.15. Let κ be a cardinal, \mathcal{A} a family of subsets of $\mathcal{P}_\kappa\lambda$ and \vec{d} a $\mathcal{P}_\kappa\lambda$ -list. Then the following are equivalent:

- (1) \vec{d} is a $\text{Split}_\kappa(\mathcal{A}, \text{nonempty})$ -sequence.
- (2) \vec{d} has no \mathcal{A} branch.

Thus, a regular, uncountable cardinal κ is \mathcal{A} -ineffable if and only if $\text{Split}_\kappa(\mathcal{A}, \text{nonempty})$ fails. In particular, κ is ineffable if and only if $\text{Split}_\kappa(\text{stationary}, \text{nonempty})$ fails, and κ is almost ineffable if and only if $\text{Split}_\kappa(\text{unbounded}, \text{nonempty})$ fails.

Proof. 1. \implies 2.: Suppose B is an \mathcal{A} branch for \vec{d} . Let $A \in \mathcal{A}$ be such that for all $\alpha \in A$, $d_\alpha = B \cap \alpha$. Let β be such that both A_β^+ and A_β^- are nonempty. Let $\gamma \in A_\beta^+$ and $\delta \in A_\beta^-$. Then $\beta \in \gamma$, $\gamma \in A$, $\beta \in d_\gamma$ and $d_\gamma = B \cap \gamma$, so $\beta \in B$. On the other hand, $\beta \in \delta$, $\delta \in A$, $\beta \notin d_\delta$ and $d_\delta = B \cap \delta$, so $\beta \notin B$. This is a contradiction.

2. \implies 1.: We show the contrapositive. So assuming \vec{d} is not a $\text{Split}_\kappa(\mathcal{A}, \text{nonempty})$ -sequence, we have to show that it has an \mathcal{A} branch. Let $A \in \mathcal{A}$ be such that no β splits A into nonempty sets.

So for every $\beta < \kappa$, it's not the case that both A_β^+ and A_β^- are nonempty. Set

$$B = \{\beta < \kappa \mid A_\beta^+ \neq \emptyset\}.$$

It follows that for every $\alpha \in A$, $d_\alpha = B \cap \alpha$ (and hence that B is an \mathcal{A} branch). To see this, let $\alpha \in A$, and let $\beta < \alpha$. If $\beta \in B$, then $A_\beta^+ \neq \emptyset$, so $A_\beta^- = \emptyset$. It follows that $\beta \in d_\alpha$ (because if we had $\beta \notin d_\alpha$, it would follow that $\alpha \in A_\beta^-$). And if $\beta \notin B$, then $A_\beta^+ = \emptyset$. Since $\beta \in \alpha$ and $\alpha \in A$, it follows that $\beta \notin d_\alpha$, because if we had $\beta \in d_\alpha$, then it would follow that $\alpha \in A_\beta^+ = \emptyset$. This shows that $d_\alpha = B \cap \alpha$, as claimed. \square

This general theorem on the failure of splitting into nonempty sets allows us to characterize subtle cardinals as well, after introducing an additional concept.

Definition 2.16. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$ be families of subsets of κ , and let $A \subseteq \kappa$ be a fixed subset of κ . Then we write $\mathcal{A} \upharpoonright A$ for $\mathcal{A} \cap \mathcal{P}(A)$, i.e., for the family of sets in \mathcal{A} that are contained in A . We let

$$\mathcal{I}(\text{Split}_\kappa(\mathcal{A}, \mathcal{B})) = \{A \subseteq \kappa \mid \text{Split}_\kappa(\mathcal{A} \upharpoonright A, \mathcal{B}) \text{ holds}\}$$

Note that if a κ -list \vec{d} witnesses that $\text{Split}_\kappa(\mathcal{A} \upharpoonright A, \mathcal{B})$ holds, then the values of d_α for $\alpha \in \kappa \setminus A$ are irrelevant, and hence it makes sense to restrict \vec{d} to A , call it an A -list, and view $\text{Split}_\kappa(\mathcal{A} \upharpoonright A, \mathcal{B})$ as postulating the existence of a splitting A -list. Note also that if $A \subseteq B \subseteq \kappa$ and $\text{Split}_\kappa(\mathcal{A} \upharpoonright B, \mathcal{B})$ holds, then so does $\text{Split}_\kappa(\mathcal{A} \upharpoonright A, \mathcal{B})$, since $\mathcal{A} \upharpoonright A \subseteq \mathcal{A} \upharpoonright B$. If $I \subseteq \mathcal{P}(X)$ is an ideal, then we write I^+ for the collection of I -positive sets, i.e., $\mathcal{P}(X) \setminus I$. We write I^* for the dual filter associated to I , which consists of the complements of sets in I .

Observation 2.17. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$ be families of subsets of κ . Suppose that $\mathcal{A} = I^+$, for some ideal I on κ , and that \mathcal{B} is closed under supersets. Then $\mathcal{I}(\text{Split}_\kappa(\mathcal{A}, \mathcal{B}))$ is an ideal.

Proof. We have already pointed out that $\mathcal{I}(\text{Split}_\kappa(\mathcal{A}, \mathcal{B}))$ is closed under subsets. Now suppose $X, Y \in \mathcal{I}(\text{Split}_\kappa(\mathcal{A}, \mathcal{B}))$. Let \vec{d}, \vec{e} be X, Y -lists witnessing that $\text{Split}_\kappa(\mathcal{A} \upharpoonright X, \mathcal{B})$, $\text{Split}_\kappa(\mathcal{A} \upharpoonright Y, \mathcal{B})$ holds, respectively. Let $\vec{f} = \vec{d} \cup \vec{e} \upharpoonright (Y \setminus X)$. Then \vec{f} is a $\text{Split}_\kappa(\mathcal{A} \upharpoonright (X \cup Y), \mathcal{B})$ -list: let $Z \in \mathcal{A}$, $Z \subseteq X \cup Y$. Then it must be the case that $Z \cap X$ or $Z \cap Y$ is in I^+ , because otherwise $Z = (Z \cap X) \cup (Z \cap Y)$ would be the union of two members of I . If $Z \cap X \in I^+$, then $Z \cap X \in \mathcal{A} \upharpoonright X$, and so, there is a $\beta < \kappa$ such that both $(Z \cap X)_{\beta, \vec{d}}^+$ and $(Z \cap X)_{\beta, \vec{d}}^-$ are in \mathcal{B} . Since $\vec{f} \upharpoonright X = \vec{d}$, it follows that $(Z \cap X)_{\vec{d}, \beta}^+ \subseteq Z_{\beta, \vec{e}}^+$ and $(Z \cap X)_{\beta, \vec{d}}^- \subseteq Z_{\beta, \vec{e}}^-$, so, since \mathcal{B} is closed under supersets, $Z_{\beta, \vec{e}}^+$ and $Z_{\beta, \vec{e}}^-$ are in \mathcal{B} . If $Z \cap X$ is in I , then $Z \cap (Y \setminus X) \in I^+$, then we can argue similarly, replacing $Z \cap X$ with $Z \cap (Y \setminus X)$. This means that I is closed under unions and is thus an ideal as desired. \square

We will explore split ideals more in the $\mathcal{P}_\kappa \lambda$ -context, in Section 4. Let's now characterize when κ is subtle. Recall that by definition, a regular cardinal κ is subtle iff for every κ -list \vec{d} and every club $C \subseteq \kappa$, there are $\alpha < \beta$ in C such that $d_\alpha = d_\beta \cap \alpha$.

Lemma 2.18. A regular cardinal is subtle iff $\mathcal{I}(\text{Split}_\kappa([\kappa]^2, \text{nonempty}))$ contains no club, i.e., iff for every club $C \subseteq \kappa$, $\text{Split}_\kappa([\kappa]^2 \upharpoonright C, \text{nonempty})$ fails.

Proof. Let κ be subtle, and suppose $C\text{-Split}_\kappa([\kappa]^2, \text{nonempty})$ held for some club $C \subseteq \kappa$. Let \vec{d} be a C -list witnessing this. The proof of Theorem 2.15 then relativizes to C and shows that \vec{d} has no $[\kappa]^2 \upharpoonright C$ branch. This means that there is no two-element subset of C on which \vec{d} coheres, contradicting our assumption that κ is subtle.

Vice versa, if κ is not subtle, then there is a club C and a C -list \vec{d} such that \vec{d} does not cohere on any two-element subset of C , i.e., \vec{d} has no $[\kappa]^2 \restriction C$ -branch, and again, by (a relativized version of) Theorem 2.15, this is equivalent to $\text{Split}_\kappa([\kappa]^2 \restriction C, \text{nonempty})$. \square

Note that the family of nonempty subsets of κ is not independent of initial segments, and so Observation 2.3 does not apply, and we do not know that $\text{Split}_\kappa(\mathcal{A}, \text{nonempty})$ is equivalent to $\text{Split}_{\kappa, \kappa}(\mathcal{A}, \text{nonempty})$. The latter is equivalent to $\mathfrak{s}_{\mathcal{A}, \text{nonempty}}(\kappa) > \kappa$. If every $A \in \mathcal{A}$ has at least two elements, then $\text{Split}_{\kappa, \kappa}(\mathcal{A}, \text{nonempty})$ holds, as witnessed by the $(\mathcal{A}, \text{nonempty})$ -splitting family $\{\{\alpha\} \mid \alpha \in \kappa\}$, while $\text{Split}_\kappa(\mathcal{A}, \text{nonempty})$ characterizes the failure of κ being \mathcal{A} -ineffable. The following theorem will allow us to characterize ineffability by split principles that can be expressed as statements about splitting numbers.

Theorem 2.19. *Let κ be a regular cardinal. The following are equivalent.*

- (1) κ is ineffable.
- (2) $\text{Split}_\kappa(\text{stationary}, \text{nonempty})$ fails.
- (3) $\text{Split}_\kappa(\text{stationary}, \text{unbounded})$ fails. Equivalently, $\mathfrak{s}_{\text{stationary}, \text{unbounded}}(\kappa) > \kappa$.
- (4) $\text{Split}_\kappa(\text{stationary})$ fails. Equivalently, $\mathfrak{s}_{\text{stationary}}(\kappa) > \kappa$.

It follows that for any family $\mathcal{B} \subseteq \mathcal{P}(\kappa)$ satisfying $\text{stationary} \subseteq \mathcal{B} \subseteq \text{nonempty}$, the failure of $\text{Split}_\kappa(\text{stationary}, \mathcal{B})$ characterizes ineffability.

Proof. 1. \iff 2. follows from Theorem 2.15.

2. \implies 3. \implies 4. is trivial

For 4. \implies 1., let \vec{d} be a κ -list. We have to find an ineffable branch. Since \vec{d} is not a $\text{Split}_\kappa(\text{stationary})$ sequence, it follows that there is a stationary set $S \subseteq \kappa$ such that for no $\beta < \kappa$ do we have that both $S_\beta^+ = S_{\beta, \vec{d}}^+$ and $S_\beta^- = S_{\beta, \vec{d}}^-$ are stationary in κ . But clearly, one of them is. For each $\beta < \kappa$ let C_β be club in κ and disjoint from the nonstationary one of S_β^+ and S_β^- . Let $C = \Delta_{\beta < \kappa} C_\beta$. Let $T = S \cap C$. Then \vec{d} coheres on the stationary set T , because if $\gamma < \delta$ both are members of T , then for $\xi < \gamma$, it follows that $\gamma, \delta \in C_\xi$, so if S_ξ^- is nonstationary, then $\gamma, \delta \in S_\xi^+$, which means that $\xi \in d_\gamma$ and $\xi \in d_\delta$. And if S_ξ^+ is nonstationary, then $\gamma, \delta \in S_\xi^-$ and it follows that $\xi \notin d_\gamma$ and $\xi \notin d_\delta$. So $d_\gamma = d_\delta \cap \gamma$. Thus, $b = \bigcup_{\alpha \in T} d_\alpha$ is an ineffable branch.

The statements about the splitting numbers follow because the families we are splitting into are independent of initial segments. \square

3. SPLITTING SUBSETS OF $\mathcal{P}_\kappa \lambda$

We may generalize the split principles to the context of $\mathcal{P}_\kappa \lambda$ -lists, and assume for the present section that κ is regular and that $\lambda > \kappa$ is a cardinal. $\mathcal{P}_\kappa \lambda$ -lists are sequences of the form $\langle d_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$ satisfying $d_x \subseteq x$ for each $x \in \mathcal{P}_\kappa \lambda$.

We use Jech's approach to stationary subsets of $\mathcal{P}_\kappa \lambda$. Thus, a set $U \subseteq \mathcal{P}_\kappa \lambda$ is called *unbounded* if for every $x \in \mathcal{P}_\kappa \lambda$, there is a $y \in U$ with $x \subseteq y$. A set $C \subseteq \mathcal{P}_\kappa \lambda$ is *club* if it is unbounded and closed under unions of increasing chains of length less than κ , and a set $S \subseteq \mathcal{P}_\kappa \lambda$ is *stationary* iff it intersects every club subset of $\mathcal{P}_\kappa \lambda$, see [Jec03, Def. 8.21]. If $f : [\lambda]^n \rightarrow \mathcal{P}_\kappa \lambda$, for some $n < \omega$, or $f : [\lambda]^{<\omega} \rightarrow \mathcal{P}_\kappa \lambda$, then we write \mathcal{C}_f for the set $\{x \in \mathcal{P}_\kappa \lambda \mid \forall a \in [x]^{<\omega} \cap \text{dom}(f) \quad f(a) \subseteq x\}$. It was shown by Menas that for every club subset C of $\mathcal{P}_\kappa \lambda$, there is a function $f : [\lambda]^2 \rightarrow \mathcal{P}_\kappa \lambda$ such that $\mathcal{C}_f \setminus \{\emptyset\} \subseteq C$. Since \mathcal{C}_f is itself club, the club filter on $\mathcal{P}_\kappa \lambda$ is generated by the sets of the form \mathcal{C}_f , and as a result, a subset S of $\mathcal{P}_\kappa \lambda$ is stationary iff it intersects every set of the form \mathcal{C}_f .

Definition 3.1. Let κ be regular and $\lambda > \kappa$ be a cardinal. Let \mathcal{A} and \mathcal{B} be families of subsets of $\mathcal{P}_\kappa\lambda$. The principle $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A}, \mathcal{B})$ says that there is a $\mathcal{P}_\kappa\lambda$ -list $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ that splits \mathcal{A} into \mathcal{B} , meaning that for every $A \in \mathcal{A}$, there is a $\beta < \lambda$ such that both $A_{\beta, \vec{d}}^+ = A_\beta^+ = \{x \in A \mid \beta \in d_x\}$ (note that $\beta \in d_x \implies \beta \in x$) and $A_{\beta, \vec{d}}^- = A_\beta^- = \{x \in A \mid \beta \in x \setminus d_x\}$ are in \mathcal{B} . We write $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A})$ for $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A}, \mathcal{A})$.

Following Carr's notation, for $x \in \mathcal{P}_\kappa\lambda$, let $\hat{x} = \{y \in \mathcal{P}_\kappa\lambda \mid x \subseteq y\}$.

We will also use generalized two-cardinal versions of the split principles, where we do not insist that the sequences are $\mathcal{P}_\kappa\lambda$ -lists, as follows. As with the original split principle, these have a close connection to splitting families and splitting numbers.

Definition 3.2. Let κ be regular, $\lambda > \kappa$ a cardinal, and τ a cardinal. Let \mathcal{A} and \mathcal{B} be families of subsets of $\mathcal{P}_\kappa\lambda$. Define $\text{Split}_{\mathcal{P}_\kappa\lambda, \tau}(\mathcal{A}, \mathcal{B})$ as before, i.e., it says that there is a sequence $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ of subsets of τ that splits \mathcal{A} into \mathcal{B} , meaning that for every $A \in \mathcal{A}$, there is a $\beta < \tau$ such that both $\tilde{A}_\beta^+ = \tilde{A}_{\beta, \vec{d}}^+ = \{x \in A \mid \beta \in d_x\}$ and $\tilde{A}_\beta^- = \tilde{A}_{\beta, \vec{d}}^- = \{x \in A \mid \beta \notin d_x\}$ are in \mathcal{B} .

Similarly, define that a collection \mathcal{S} of subsets of $\mathcal{P}_\kappa\lambda$ is an $(\mathcal{A}, \mathcal{B})$ -splitting family for $\mathcal{P}_\kappa\lambda$, if for every $A \in \mathcal{A}$ there is an $S \in \mathcal{S}$ such that $A \cap S$ and $A \setminus S$ are in \mathcal{B} (\mathcal{S} splits \mathcal{A} into \mathcal{B}). The splitting number $\mathfrak{s}_{\mathcal{A}, \mathcal{B}}(\mathcal{P}_\kappa\lambda)$ is the smallest cardinality of an $(\mathcal{A}, \mathcal{B})$ -splitting family.

The following lemma says that the generalized split principles on $\mathcal{P}_\kappa\lambda$ can be viewed as statements about the sizes of the corresponding splitting numbers.

Lemma 3.3. Let $\kappa < \lambda$ be cardinals, τ a cardinal, and let \mathcal{A}, \mathcal{B} be families of subsets of $\mathcal{P}_\kappa\lambda$. Then $\text{Split}_{\mathcal{P}_\kappa\lambda, \tau}(\mathcal{A}, \mathcal{B})$ holds iff $\mathfrak{s}_{\mathcal{A}, \mathcal{B}}(\mathcal{P}_\kappa\lambda) \leq \tau$.

Note: In other words, $\mathfrak{s}_{\mathcal{A}, \mathcal{B}}(\mathcal{P}_\kappa\lambda)$ is the least τ such that $\text{Split}_{\mathcal{P}_\kappa\lambda, \tau}(\mathcal{A}, \mathcal{B})$ holds.

Proof. For the direction from right to left, if $\mathcal{S} = \{S_\alpha \mid \alpha < \tau\}$ is an $(\mathcal{A}, \mathcal{B})$ -splitting family for $\mathcal{P}_\kappa\lambda$, then we can define a sequence $\langle d_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ of subsets of τ by setting

$$d_x = \{\gamma < \tau \mid x \in S_\gamma\}.$$

Then \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda, \tau}(\mathcal{A}, \mathcal{B})$ sequence, because if $A \in \mathcal{A}$, then there is a $\beta < \tau$ such that both $A \cap S_\beta$ and $A \setminus S_\beta$ belong to \mathcal{B} , but $A \cap S_\beta = \tilde{A}_{\beta, \vec{d}}^+$ and $A \setminus S_\beta = \tilde{A}_{\beta, \vec{d}}^-$ so we are done.

Conversely, if \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda, \tau}(\mathcal{A}, \mathcal{B})$ sequence, then for each $\gamma < \tau$, we define a subset S_γ of $\mathcal{P}_\kappa\lambda$ by

$$S_\gamma = \{x \in \mathcal{P}_\kappa\lambda \mid \gamma \in d_x\}.$$

Then $\mathcal{S} = \{S_\gamma \mid \gamma < \tau\}$ is an $(\mathcal{A}, \mathcal{B})$ -splitting family for $\mathcal{P}_\kappa\lambda$, because if $A \in \mathcal{A}$, then there is a $\beta < \tau$ such that both $\tilde{A}_{\beta, \vec{d}}^+$ and $\tilde{A}_{\beta, \vec{d}}^-$ are in \mathcal{B} , but as before, $\tilde{A}_{\beta, \vec{d}}^+ = A \cap S_\beta$ and $\tilde{A}_{\beta, \vec{d}}^- = A \setminus S_\beta$. \square

The following are the families of subsets of $\mathcal{P}_\kappa\lambda$ we will mostly be working with as our collections \mathcal{A} and \mathcal{B} with split principles and splitting numbers.

Definition 3.4.

- **unbounded** is the set of unbounded subsets of $\mathcal{P}_\kappa\lambda$ (note that “unbounded” is not the same as “not bounded” - the more correct term would be “cofinal”, but “unbounded” is the commonly accepted term. So $U \subseteq \mathcal{P}_\kappa\lambda$ is unbounded iff for every $x \in \mathcal{P}_\kappa\lambda$ there is a $y \in \mathcal{P}_\kappa\lambda$ with $x \subseteq y$.)

- **covering** is the set of $A \subseteq \mathcal{P}_\kappa \lambda$ such that $\cup A = \lambda$, i.e., for every $\xi < \lambda$, there is an $x \in A$ with $\xi \in x$ (note that in the space κ rather instead of $\mathcal{P}_\kappa \lambda$, not bounded, unbounded and covering are equivalent).
- **stationary** is the collection of stationary subsets of $\mathcal{P}_\kappa \lambda$.
- **nonempty** is the collection of nonempty subsets of $\mathcal{P}_\kappa \lambda$.

Observation 3.5. *Let $\tau \leq \kappa \leq \lambda$ be cardinals, and let \mathcal{A} and \mathcal{B} be collections of subsets of κ such that for all $B \subseteq \kappa$ and all $\beta < \tau$, $B \in \mathcal{B}$ iff $B \cap \widehat{\{\beta\}} \in \mathcal{B}$ (“ \mathcal{B} is independent of initial segments”). Then $\text{Split}_{\mathcal{P}_\kappa \lambda, \tau}(\mathcal{A}, \mathcal{B})$ is equivalent to the existence of a $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \mathcal{B})$ -sequence \vec{d} of subsets of τ .*

Proof. If \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa \lambda, \tau}(\mathcal{A}, \mathcal{B})$ -sequence, then the $\mathcal{P}_\kappa \lambda$ -list \vec{e} defined by $e_x = d_x \cap x$ is a sequence of subsets of τ that is a $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \mathcal{B})$ -sequence. This is because if $A \in \mathcal{A}$ and $\beta < \tau$ is such that $\tilde{A}_{\beta, \vec{d}}^+$ and $\tilde{A}_{\beta, \vec{d}}^-$ are in \mathcal{B} , then $A_{\beta, \vec{e}}^+ \cap \widehat{\{\beta\}} = \tilde{A}_{\beta, \vec{d}}^+ \cap \widehat{\{\beta\}}$, and similarly, $A_{\beta, \vec{e}}^- \cap \widehat{\{\beta\}} = \tilde{A}_{\beta, \vec{d}}^- \cap \widehat{\{\beta\}}$. It follows from our assumption on \mathcal{B} that $A_{\beta, \vec{e}}^+$ and $A_{\beta, \vec{e}}^-$ are in \mathcal{B} . \square

Note that **unbounded** and **stationary** are independent of initial segments, while **covering** and **nonempty** are not.

Observation 3.6. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$, and let \mathcal{B} be closed under supersets (i.e., if $B \in \mathcal{B}$ and $B \subseteq C \subseteq \mathcal{P}_\kappa \lambda$, then $C \in \mathcal{B}$). Then every $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \mathcal{B})$ -sequence is a $\text{Split}_{\mathcal{P}_\kappa \lambda, \lambda}(\mathcal{A}, \mathcal{B})$ -sequence.*

Proof. Let \vec{d} be a $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \mathcal{B})$ -sequence. For any $A \in \mathcal{A}$ and any $\beta < \lambda$, $A_{\vec{d}, \beta}^+ = \tilde{A}_{\vec{d}, \beta}^+$ and $A_{\vec{d}, \beta}^- \subseteq \tilde{A}_{\vec{d}, \beta}^-$. So since \mathcal{B} is closed under supersets, it follows that \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa \lambda, \lambda}(\mathcal{A}, \mathcal{B})$ -sequence. \square

Observation 3.7. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\kappa)$ such that $\mathcal{B} \subseteq \text{covering}$. Then the principle $\text{Split}_{\mathcal{P}_\kappa \lambda, \lambda}(\mathcal{A}, \mathcal{B})$ implies $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \text{nonempty})$.*

Proof. Let \vec{d} be a $\text{Split}_{\mathcal{P}_\kappa \lambda, \lambda}(\mathcal{A}, \mathcal{B})$ sequence. Let $e_x = d_x \cap x$, for $x \in \mathcal{P}_\kappa \lambda$. We show that \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \text{nonempty})$ sequence: let $A \in \mathcal{A}$. Let $\beta < \lambda$ be such that \tilde{A}_β^+ and \tilde{A}_β^- both are in \mathcal{B} . In particular, both of these sets are covering. So let $x \in \tilde{A}_\beta^+$ and let $y \in \tilde{A}_\beta^-$. Then $x \in A$, and $\beta \in d_x \cap x = e_x$, so $x \in A_{\beta, \vec{e}}^+$, and $y \in A$, and $\beta \in y \setminus d_y = y \setminus e_y$, so $y \in A_{\beta, \vec{e}}^-$. \square

The terminology introduced in the next definition follows [Wei10]. It is the concept corresponding to sequential binary trees from the previous section.

Definition 3.8. A *forest* on $\mathcal{P}_\kappa \lambda$ is a set $\mathcal{F} \subseteq {}^{\mathcal{P}_\kappa \lambda} 2$ such that for every $f \in \mathcal{F}$, if $x \subseteq \text{dom}(f)$, then $f \restriction x \in \mathcal{F}$, and such that for every $x \in \mathcal{P}_\kappa \lambda$, there is an $f \in \mathcal{F}$ such that $x = \text{dom}(f)$. A cofinal branch through \mathcal{F} is a function $B : \lambda \rightarrow 2$ such that for every $x \in \mathcal{P}_\kappa \lambda$, $B \restriction x \in \mathcal{F}$.

In [Jec73], forests were referred to as *binary* (κ, λ) -*messes*, and cofinal branches through forests were called *solutions to binary messes*. Clearly, there is an obvious way to assign forests to lists.

Definition 3.9. Given a $\mathcal{P}_\kappa \lambda$ -list $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$, for each $x \in \mathcal{P}_\kappa \lambda$ let $d_x^c : x \rightarrow 2$ denote the characteristic function of d_x .

By closing these characteristic functions downward, we may consider the forest $\mathcal{F}_{\vec{d}}$ corresponding to the $\mathcal{P}_\kappa \lambda$ -list \vec{d} , defined by $\mathcal{F}_{\vec{d}} = \{d_x^c \restriction y \mid y \subseteq x \in \mathcal{P}_\kappa \lambda\}$.

Below we define several types of branches for $\mathcal{P}_\kappa \lambda$ -lists, similar to our treatment of κ -lists. Some of our terminology is inspired by work of DiPrisco, Zwicker and Carr.

Definition 3.10. Let $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$ be a $\mathcal{P}_\kappa \lambda$ -list.

A set $B \subseteq \lambda$ is a *cofinal branch* through \vec{d} so long as for all $x \in \mathcal{P}_\kappa \lambda$, there is some $y \in \mathcal{P}_\kappa \lambda$ with $y \supseteq x$ such that $d_y \cap x = B \cap x$. The *branch property* $\text{BP}(\kappa, \lambda)$ holds iff every $\mathcal{P}_\kappa \lambda$ -list has a cofinal branch, and κ is *mildly λ -ineffable* iff $\text{BP}(\kappa, \lambda)$ holds (the origin of mild ineffability is [DZ80]).

A cofinal branch B is *guided* by a set $U \subseteq \mathcal{P}_\kappa \lambda$ if for all $x \in \mathcal{P}_\kappa \lambda$ there is a $y \supseteq x$ such that for all $z \supseteq y$ with $z \in U$, $d_z \cap x = B \cap x$. A cofinal branch B is *strong* if it is guided by an unbounded set. If every $\mathcal{P}_\kappa \lambda$ -list has a strong branch, then the *strong branch property* $\text{SBP}(\kappa, \lambda)$ holds, and we say that κ is *wildly λ -ineffable*.

An *almost ineffable branch* through \vec{d} is a subset $B \subseteq \lambda$ such that there is an unbounded set $U \subseteq \mathcal{P}_\kappa \lambda$ such that for all $x \in U$, $d_x = B \cap x$. If every $\mathcal{P}_\kappa \lambda$ -list has an almost ineffable branch, then $\text{AIBP}(\kappa, \lambda)$ holds, and κ is *almost λ -ineffable*.

An *ineffable branch* through \vec{d} is a subset $B \subseteq \lambda$ which comes with a stationary set $S \subseteq \mathcal{P}_\kappa \lambda$ such that for all $x \in S$, $d_x = B \cap x$. If every $\mathcal{P}_\kappa \lambda$ -list has an ineffable branch, then $\text{IBP}(\kappa, \lambda)$ holds, and κ is *λ -ineffable*.

In general, if \mathcal{A} is a family of subsets of $\mathcal{P}_\kappa \lambda$ and $B \subseteq \lambda$, then we say that B is an \mathcal{A} branch of \vec{d} if there is a set $A \in \mathcal{A}$ such that for all $x \in A$, $d_x = B \cap x$.

If $\mathcal{F} = \mathcal{F}_{\vec{d}}$ is the forest corresponding to \vec{d} , then we refer to a function $B : \lambda \rightarrow 2$ as a (cofinal, strong, almost ineffable, ineffable) branch of \mathcal{F} if the set $\{\alpha < \lambda \mid B(\alpha) = 1\}$ is a (cofinal, strong, almost ineffable, ineffable) branch through \vec{d} .

Observation 3.11. Let \vec{d} be a $\mathcal{P}_\kappa \lambda$ -list, and let $B \subseteq \lambda$. If B is an ineffable branch then it is an almost ineffable branch. If it is an almost ineffable branch, then it is a strong branch. If it is a strong branch, then it is a cofinal branch. So we have the following string of implications: $\text{IBP}(\kappa, \lambda) \implies \text{AIBP}(\kappa, \lambda) \implies \text{SBP}(\kappa, \lambda) \implies \text{BP}(\kappa, \lambda)$.

3.1. Split characterizations of two cardinal versions of ineffability. The next goal is to establish characterizations of the classical two cardinal versions of ineffability and almost ineffability. First, let us state a very general theorem.

Theorem 3.12. Let $\kappa \leq \lambda$ be cardinals, \mathcal{A} a family of subsets of $\mathcal{P}_\kappa \lambda$ and \vec{d} a $\mathcal{P}_\kappa \lambda$ -list. Then the following are equivalent:

- (1) \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \text{nonempty})$ -sequence.
- (2) \vec{d} has no \mathcal{A} branch.

Proof. 1. \implies 2.: Suppose B is an \mathcal{A} branch for \vec{d} . Let $A \in \mathcal{A}$ be such that for all $x \in A$, $d_x = B \cap x$. Let $\beta < \lambda$ be such that both A_β^+ and A_β^- are nonempty. Let $x \in A_\beta^+$ and $y \in A_\beta^-$. Then $\beta \in x$, $x \in A$, $\beta \in d_x$ and $d_x = B \cap x$, so $\beta \in B$. On the other hand, $\beta \in y$, $y \in A$, $\beta \notin d_y$ and $d_y = B \cap y$, so $\beta \notin B$. This is a contradiction.

2. \implies 1.: We show the contrapositive. So assuming \vec{d} is not a $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{A}, \text{nonempty})$ -sequence, we have to show that it has an \mathcal{A} branch. Let $A \in \mathcal{A}$ be such that no $\beta < \lambda$ splits A into nonempty sets. So for every $\beta < \lambda$, it's not the case that both A_β^+ and A_β^- are nonempty. Set

$$B = \{\beta < \lambda \mid A_\beta^+ \neq \emptyset\}$$

It follows that for every $x \in A$, $d_x = B \cap x$ (and hence that B is an \mathcal{A} branch). To see this, let $x \in A$, and let $\beta \in x$. If $\beta \in B$, then $A_\beta^+ \neq \emptyset$, so $A_\beta^- = \emptyset$. It follows that $\beta \in d_x$ (because if we had $\beta \notin d_x$, it would follow that $x \in A_\beta^-$). And if $\beta \notin B$, then $A_\beta^+ = \emptyset$. Since $\beta \in x$ and $x \in A$, it

follows that $\beta \notin d_x$, because if we had $\beta \in d_x$, then it would follow that $x \in A_\beta^+ = \emptyset$. This shows that $d_x = B \cap x$, as claimed. \square

The following lemma is an immediate consequence.

Lemma 3.13. *Let $\kappa \leq \lambda$.*

- (1) *κ is almost λ -ineffable iff $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}, \text{nonempty})$ fails.*
- (2) *κ is λ -ineffable iff $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{nonempty})$ fails.*

It turns out that the characterization of ineffability by the failure of split principles of the form $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \mathcal{B})$ is very robust.

Theorem 3.14. *Let κ be regular and uncountable, and $\lambda \geq \kappa$ be a cardinal. Let \vec{d} be a $\mathcal{P}_\kappa\lambda$ -list. The following are equivalent:*

- (1) *\vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary})$ sequence.*
- (2) *\vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{unbounded})$ sequence.*
- (3) *\vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{covering})$ sequence.*
- (4) *\vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{nonempty})$ sequence.*
- (5) *\vec{d} has no ineffable branch.*

In general, these are equivalent to saying that \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \mathcal{B})$ sequence whenever $\text{stationary} \subseteq \mathcal{B} \subseteq \text{nonempty}$.

Proof. 1. \implies 2. \implies 3. \implies 4. is immediate, because every stationary set is unbounded, every unbounded set is covering, and every covering set is nonempty.

4. \implies 5. follows from Theorem 3.12.

For 5. \implies 1., we prove the contrapositive, i.e., assuming \vec{d} does not split stationary sets into stationary sets, we show that \vec{d} has an ineffable branch. Let $S \subseteq \mathcal{P}_\kappa\lambda$ be a stationary set such that for each $\beta < \lambda$, not both S_β^+ and S_β^- are stationary in $\mathcal{P}_\kappa\lambda$. Since $S \cap \widehat{\{\beta\}} = S_\beta^+ \cup S_\beta^-$, this means that exactly one of them is stationary (for each β). Define

$$B = \{\beta < \lambda \mid S_\beta^+ \text{ is stationary}\}.$$

Let C_β be club in $\mathcal{P}_\kappa\lambda$ and disjoint from the nonstationary one of S_β^+ and S_β^- . Let $D = \triangle_{\beta < \lambda} C_\beta = \{x \in \mathcal{P}_\kappa\lambda \mid \forall \beta \in x \ x \in C_\beta\}$. Then D is club, and so, $E = S \cap D$ is stationary. But \vec{d} coheres with B on E : let $x \in E$. We have to show that $B \cap x = d_x$. So let $\beta \in x$. Then $x \in C_\beta$. If $\beta \in B$, then S_β^+ is stationary, so $C_\beta \cap S_\beta^- = \emptyset$, and so, $x \in S_\beta^+$ (since $\beta \in x$), which means that $\beta \in d_x$. And if $\beta \notin B$, then S_β^- is stationary, and it follows that $x \in S_\beta^-$, so $\beta \notin d_x$. This shows that B is an ineffable branch. \square

As an immediate consequence, we get:

Lemma 3.15. *If κ is regular and uncountable, and $\lambda \geq \kappa$ is a cardinal, then the following are equivalent:*

- (1) *κ is λ -ineffable.*
- (2) *$\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary})$ fails.*
- (3) *$\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{unbounded})$ fails.*
- (4) *$\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{covering})$ fails.*
- (5) *$\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \text{nonempty})$ fails.*

In general, these are equivalent to saying that $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{stationary}, \mathcal{B})$ fails whenever $\text{stationary} \subseteq \mathcal{B} \subseteq \text{nonempty}$.

The previous two facts go through in more generality:

Theorem 3.16. *Let κ be regular and uncountable, and $\lambda \geq \kappa$ be a cardinal. Let \vec{d} be a $\mathcal{P}_\kappa\lambda$ -list, and let \mathcal{I} be a normal ideal on $\mathcal{P}_\kappa\lambda$. The following are equivalent:*

- (1) \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{I}^+)$ sequence.
- (2) \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{I}^+, \text{nonempty})$ sequence.
- (3) \vec{d} has no \mathcal{I}^+ branch.

In general, these are equivalent to saying that \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{I}^+, \mathcal{B})$ sequence whenever $\mathcal{I}^+ \subseteq \mathcal{B} \subseteq \text{nonempty}$.

Proof. 1. \implies 2. is immediate, and 2. \implies 3. follows from Theorem 3.12.

For 3. \implies 1., we prove the contrapositive, i.e., assuming \vec{d} does not split \mathcal{I} -positive sets into sets in \mathcal{I}^+ , we show that \vec{d} has an \mathcal{I}^+ branch. Let $S \in \mathcal{I}^+$ be such that for each $\beta < \lambda$, not both S_β^+ and S_β^- are in \mathcal{I}^+ . Note that $\widehat{\{\beta\}} \in \mathcal{I}^*$: this is because $\widehat{\{\beta\}}$ is club, and the club filter is the minimal normal filter (see [Car81]), and \mathcal{I}^* is a normal filter. It follows that for each $\beta < \kappa$, $S \cap \widehat{\{\beta\}} \in \mathcal{I}^+$ (this is equivalent to saying that $S \cap \widehat{\{\beta\}} \notin \mathcal{I}$). To see this, suppose instead we had $S \cap \widehat{\{\beta\}} \in \mathcal{I}$. As $\mathcal{P}_\kappa\lambda \setminus \widehat{\{\beta\}} \in \mathcal{I}$ also $(S \cap \widehat{\{\beta\}}) \cup (\mathcal{P}_\kappa\lambda \setminus \widehat{\{\beta\}}) \in \mathcal{I}$, but S is a subset of that, so $S \in \mathcal{I}$, a contradiction.

Since $S \cap \widehat{\{\beta\}} = S_\beta^+ \cup S_\beta^-$, this means that exactly one of S_β^+ , S_β^- is in \mathcal{I}^+ (for each β) - we know that not both of them are in \mathcal{I}^+ . If neither of them were in \mathcal{I}^+ , then both would be in \mathcal{I} , but then their union would also be in \mathcal{I} .

Define

$$B = \{\beta < \lambda \mid S_\beta^+ \text{ is in } \mathcal{I}^+\}.$$

We show that B is an \mathcal{I} branch as follows. Let C_β be in \mathcal{I}^* and disjoint from the one of S_β^+ and S_β^- that's in \mathcal{I} . By normality, $D = \Delta_{\beta < \lambda} C_\beta = \{x \in \mathcal{P}_\kappa\lambda \mid \forall \beta \in x \ x \in C_\beta\}$ is in \mathcal{I}^* , and so, $E = S \cap D$ is in \mathcal{I}^+ . But \vec{d} coheres with B on E : let $x \in E$. We have to show that $B \cap x = d_x$. So let $\beta \in x$. Then $x \in C_\beta$. If $\beta \in B$, then S_β^+ is in \mathcal{I}^+ , $C_\beta \cap S_\beta^- = \emptyset$, and so, $x \in S_\beta^+$ (since $\beta \in x$), which means that $\beta \in d_x$. And if $\beta \notin B$, then S_β^- is in \mathcal{I}^+ , and it follows that $x \in S_\beta^-$, so $\beta \notin d_x$. This shows that B is an \mathcal{I}^+ branch. \square

It turns out that the failure of split principles of the form $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}, \mathcal{B})$ is less robust. We have seen that if $\mathcal{B} = \text{nonempty}$, the principle characterizes almost ineffability. The following theorem explores the other extreme, $\mathcal{B} = \text{unbounded}$.

Theorem 3.17. *Let \vec{d} be a $\mathcal{P}_\kappa\lambda$ -list. Then \vec{d} is a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded})$ sequence iff \vec{d} does not have a strong branch.*

Proof. We will show each direction of the implication separately.

\implies : Towards a contradiction, assume B is a strong branch through \vec{d} , guided by the unbounded set $U \subseteq \mathcal{P}_\kappa\lambda$. Now define a function f so that for all $x \in \mathcal{P}_\kappa\lambda$, $f(x) \supseteq x$, $f(x) \in U$ and for all $z \supseteq f(x)$, if $z \in U$ then $d_z \cap x = B \cap x$.

Consider the unbounded set $A = f''\mathcal{P}_\kappa\lambda \subseteq U$, and let β split A into unbounded sets, with respect to \vec{d} .

Suppose that $\beta \in B$. Now we may choose $y \in A_\beta^-$ such that $y \supseteq f(\{\beta\})$, since A_β^- is unbounded. Then $y \in U$, and by the definition of $f(\{\beta\})$ we have that $d_y \cap \{\beta\} = B \cap \{\beta\}$. Since $\beta \in B$, this means that $\beta \in d_y$, but since $y \in A_\beta^-$, $\beta \notin d_y$, a contradiction. The case $\beta \notin B$ works similarly by picking some $y \supseteq f(\{\beta\})$ with $y \in A_\beta^+$.

\Leftarrow : We will show the contrapositive. So suppose $\vec{d} = \langle d_x \mid x \in \mathcal{P}_\kappa \lambda \rangle$ does not split unbounded sets. We claim that \vec{d} has a strong branch. Since \vec{d} does not split unbounded sets, there is an unbounded $A \subseteq \mathcal{P}_\kappa \lambda$ such that for each $\beta < \lambda$, exactly one of A_β^+ and A_β^- is unbounded in $\mathcal{P}_\kappa \lambda$. Define a strong branch $B \subseteq \lambda$ by setting

$$\beta \in B \iff A_\beta^+ \text{ is unbounded in } \mathcal{P}_\kappa \lambda.$$

We claim that B is a strong branch, guided by the unbounded set A . To see this, fix $x \in \mathcal{P}_\kappa \lambda$. For each element $\beta < \lambda$ of x , note that there has to be a $y_\beta \in \mathcal{P}_\kappa \lambda$ such that either for all $z \supseteq y_\beta$, if $z \in A$ then $\beta \in d_z$; or for all $z \supseteq y_\beta$, if $z \in A$ then $\beta \notin d_z$. The former holds if A_β^- is not unbounded, and the latter holds if A_β^+ is not unbounded. Since one of those statements has to be true, y_β is defined for each $\beta < \lambda$. Let $y = \bigcup_{\beta \in x} y_\beta$. Since κ is regular, $y \in \mathcal{P}_\kappa \lambda$. Pick $z \supseteq y$ such that $z \in A$. Now $B \cap x = d_z \cap x$ as desired. \square

Corollary 3.18. *The following are equivalent:*

- (1) $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{unbounded})$ fails.
- (2) κ is wildly λ -ineffable.
- (3) Every forest on $\mathcal{P}_\kappa \lambda$ has a strong branch.

Proof. 1. and 2. are obviously equivalent. For 3. \implies 1., assume the contrary, and let \vec{d} be a $\mathcal{P}_\kappa \lambda$ -list that splits unbounded sets into unbounded sets. Then its forest $\mathcal{F}_{\vec{d}}$ does not have a strong branch, by Theorem 3.17. This contradicts the assumption that every forest on $\mathcal{P}_\kappa \lambda$ has a strong branch.

For 1. \implies 3., given a forest \mathcal{F} , for every $x \in \mathcal{P}_\kappa \lambda$ choose a function $f_x \in \mathcal{F}$ with $\text{dom}(f_x) = x$, and let $d_x = \{\gamma \in x \mid f_x(\gamma) = 1\}$. Then \vec{d} is a $\mathcal{P}_\kappa \lambda$ -list, and since $\mathcal{P}_\kappa \lambda$ -Split fails, \vec{d} does not split unbounded sets, so that by Theorem 3.17, \vec{d} has a strong branch. Letting B be the characteristic function of this strong branch, it follows that B is a strong branch of $\mathcal{F}_{\vec{d}}$, and since $\mathcal{F}_{\vec{d}} \subseteq \mathcal{F}$, B is also a strong branch of \mathcal{F} . \square

So the concept of a strong branch comes up naturally in the context of split principles. Since its exact relationship to the concept of a cofinal branch is somewhat mysterious, we want to take some time to elaborate on it. First, the existence of strong branches through $\mathcal{P}_\kappa \lambda$ -lists can be formulated as coherence properties.

Observation 3.19. *Let \vec{d} be a $\mathcal{P}_\kappa \lambda$ -list.*

- (1) \vec{d} has an ineffable branch if there is a stationary set $S \subseteq \mathcal{P}_\kappa \lambda$ such that for all $x, y \in S$ with $x \subseteq y$, $d_x = d_y \cap x$.
- (2) \vec{d} has an almost ineffable branch if there is an unbounded set $U \subseteq \mathcal{P}_\kappa \lambda$ such that for all $x, y \in U$ with $x \subseteq y$, $d_x = d_y \cap x$.
- (3) \vec{d} has a strong branch if there is an unbounded set $U \subseteq \mathcal{P}_\kappa \lambda$ such that for all $x \in \mathcal{P}_\kappa \lambda$, there is a $y \supseteq x$ such that for all $z_0, z_1 \supseteq y$ with $z_0, z_1 \in U$, $d_{z_0} \cap x = d_{z_1} \cap x$.

Thus, wild ineffability can be viewed as expressing a delayed coherence property of $\mathcal{P}_\kappa \lambda$ -lists, removing all mention of the existence of strong branches.

Definition 3.20. Let's call a function $f : \mathcal{P}_\kappa \lambda \longrightarrow \mathcal{P}_\kappa \lambda$ a *delay function* if for all $x \in \mathcal{P}_\kappa \lambda$ we have that $x \subseteq f(x)$. Let's say that a $\mathcal{P}_\kappa \lambda$ -list \vec{d} *coheres on a set* $U \subseteq \mathcal{P}_\kappa \lambda$ *with delay function* f if for all x and all $z_0, z_1 \supseteq f(x)$ with $z_0, z_1 \in U$ we have that $d_{z_0} \cap x = d_{z_1} \cap x$. If a $\mathcal{P}_\kappa \lambda$ -list coheres on U with delay function id , then let's say that it coheres immediately on U . A *continuous delay function* is a delay function f such that for some function $g : \lambda \longrightarrow \lambda$, $f(x) = x \cup g"x$.

Using this vocabulary, κ is almost λ -ineffable if every $\mathcal{P}_\kappa \lambda$ -list coheres immediately on an unbounded set, it is λ -ineffable if every $\mathcal{P}_\kappa \lambda$ -list coheres immediately on a stationary set, and it is wildly λ -ineffable if it coheres on an unbounded set with some delay function. Actually, an analysis of the proof of Theorem 3.17 shows that κ is λ -wildly ineffable, then every $\mathcal{P}_\kappa \lambda$ -list coheres on an unbounded set with a delay function of the form $f(x) = x \cup \bigcup_{\alpha \in x} g(\alpha)$, where $g : \lambda \longrightarrow \mathcal{P}_\kappa \lambda$. Observe that if \vec{d} coheres on a stationary set $S \subseteq \mathcal{P}_\kappa \lambda$ with such a delay function, then it has an ineffable branch, because the set $C_g = \{x \mid \forall \alpha \in x \quad g(\alpha) \subseteq x\}$ is club, and so, $S \cap C_g$ is stationary, but if $x \subseteq y$ with $x, y \in S \cap C_g$, then $f(x) \subseteq x \subseteq y$, so $d_x = d_x \cap x = d_y \cap x$.

We have explored $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{unbounded}, \mathcal{B})$ for $\mathcal{B} = \text{unbounded}$ and $\mathcal{B} = \text{nonempty}$. It turns out that the case $\mathcal{B} = \text{covering}$ characterizes continuously delayed coherence.

Lemma 3.21. *Let $\kappa \leq \lambda$ be cardinals, and let \vec{d} be a $\mathcal{P}_\kappa \lambda$ -list. The following are equivalent:*

- (1) $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{unbounded}, \text{covering})$.
- (2) \vec{d} does not cohere on an unbounded set with a continuous delay function.

Proof. 1. \implies 2.: Suppose there were a set $B \subseteq \lambda$ such that for some unbounded U and some $f : \lambda \longrightarrow \lambda$, we'd have that for every x and every $y \in U$ with $x \cup f"x \subseteq y$, $B \cap x = d_y \cap x$ - this is equivalent to continuously delayed coherence on an unbounded set. Let β split U into covering sets. Let $x \in U_\beta^+$, $y \in U_\beta^-$ be such that $f(\beta) \in x$, $f(\beta) \in y$. Let $a = \{\beta\}$. Then $b := a \cup f"a = \{\beta, f(\beta)\} \subseteq x \in U$, so $B \cap a = d_y \cap a$, and since $x \in U_\beta^+$, it follows that $\beta \in d_x$, so $\beta \in B$. But also, $b \subseteq y \in U$, and since $y \in U_\beta^-$, it follows that $\beta \notin d_y$, and $B \cap a = d_y \cap a$, i.e., $\beta \notin B$, a contradiction.

2. \implies 1.: We show the contrapositive. So assuming \vec{d} doesn't split unbounded sets into covering sets, we have to prove coherence with continuous delay. Let U be an unbounded set that is not split into covering sets by any $\beta < \lambda$, wrt. \vec{d} . Then for each $\beta < \lambda$, it's not the case that both U_β^+ and U_β^- cover λ . So there is an $f(\beta) < \lambda$ such that

- (a) for every $x \in U$ with $f(\beta) \in x$, $x \notin U_\beta^+$, OR
- (b) for every $x \in U$ with $f(\beta) \in x$, $x \notin U_\beta^-$.

Note that these two cases are mutually exclusive, because there is an $x \in U$ with $\{\beta, f(\beta)\} \subseteq x$, and if $x \notin U_\beta^+$, since $\beta \in x$, it follows that $\beta \notin d_x$, and so, $x \in U_\beta^-$. So exactly one of the two holds. Let

$$B = \{\beta < \lambda \mid \forall x \in U \quad f(\beta) \in x \implies x \notin U_\beta^-\}$$

We claim that for every x , and every $y \in U$ with $x \cup f"x \subseteq y$, it follows that $B \cap x = d_y \cap x$ (and this implies that \vec{d} coheres on U with delay function $x \mapsto x \cup f"x$). To see this, let $\beta \in x$. If $\beta \in B$, then it follows from the definition of B that $y \notin U_\beta^-$. But since $\beta \in x \subseteq y$ and $y \in U$, this implies that $\beta \in d_y$. On the other hand, if $\beta \notin B$, then we're not in case (b) above, so we're in case (a). So $y \in U_\beta^+$, so $\beta \in d_y$. \square

Definition 3.22. Let $f : \mathcal{P}_\kappa\lambda \rightarrow \mathcal{P}_\kappa\lambda$, and let $\vec{U} = \langle U_x \mid x \in \mathcal{P}_\kappa\lambda \rangle$ be a sequence of subsets of $\mathcal{P}_\kappa\lambda$. Define the *f-diagonal intersection* of \vec{U} as

$$\Delta^f \vec{U} = \Delta_{x \in \mathcal{P}_\kappa\lambda}^f U_x = \{z \mid \forall x (f(x) \subseteq z \implies z \in U_x)\}.$$

Observation 3.23. Suppose B is a cofinal branch for the $\mathcal{P}_\kappa\lambda$ -list \vec{d} . For $x \in \mathcal{P}_\kappa\lambda$, let

$$U_x = \{y \supseteq x \mid B \cap x = d_y \cap x\}.$$

Then B is a strong branch iff there is a delay function f such that $U = \Delta_{x \in \mathcal{P}_\kappa\lambda}^f U_x$ is unbounded.

Proof. From right to left, if f and U are as stated, then U is an unbounded set that guides B , because for any x , if $z \supseteq f(x)$ is in U , then $z \in U_x$, and so, $B \cap x = d_z \cap x$.

Vice versa, if B is strong and U' is an unbounded set that guides B , then we can define a delay function $f : \mathcal{P}_\kappa\lambda \rightarrow \mathcal{P}_\kappa\lambda$ such that for every $x \in \mathcal{P}_\kappa\lambda$, and for every $z \supseteq f(x)$ with $z \in U'$, $d_z \cap x = B \cap x$. It follows that $U' \subseteq \Delta_{x \in \mathcal{P}_\kappa\lambda}^f U_x$, because if $z \in U'$ and x is such that $f(x) \subseteq z$, then by the property of f , $d_z \cap x = B \cap x$, that is, $z \in U_x$. So since U' is unbounded, so is $\Delta^f \vec{U}$. \square

Question 3.24. Can there be a $\mathcal{P}_\kappa\lambda$ -list that has a cofinal branch but no strong branch?

Digressing briefly, we want to explore a connection to $\mathcal{P}_\kappa\lambda$ -partition properties, see [Kan03, p. 346] for an overview. For a natural number n and a subset $X \subseteq \mathcal{P}_\kappa\lambda$, write $[X]_{\subseteq}^n$ for the set

$$\{\{a_0, a_1, \dots, a_{n-1}\} \mid a_0 \subsetneq a_1 \subsetneq \dots \subsetneq a_{n-1} \in X\},$$

and say that $\text{Part}(\kappa, \lambda)^n$ holds if for every partition function $F : [\mathcal{P}_\kappa\lambda]_{\subseteq}^n \rightarrow 2$, there is an unbounded set $H \subseteq \mathcal{P}_\kappa\lambda$ that's homogeneous for F , meaning that $F \upharpoonright [H]^n$ is constant. $\text{Part}(\kappa, \lambda)$ is just $\text{Part}(\kappa, \lambda)^2$.

The following has a precursor in [Car81, Thm. 2.2].

Theorem 3.25. $\text{Part}(\kappa, \lambda)^3$ implies that every $\mathcal{P}_\kappa\lambda$ -list has a strong branch (i.e., that κ is wildly λ -ineffable, or $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded})$ fails).

Proof. The proof of [Car81, Thm. 2.2], in which it was pointed out that the assumption implies that κ is inaccessible, works here as well. Let \vec{d} be a $\mathcal{P}_\kappa\lambda$ -list, and define a partition $F : [\mathcal{P}_\kappa\lambda]_{\subseteq}^3 \rightarrow 2$ by setting, for $x \subsetneq y \subsetneq z$,

$$F(x, y, z) = \begin{cases} 0 & \text{if } d_y \cap x = d_z \cap x \\ 1 & \text{otherwise.} \end{cases}$$

Let H be an unbounded subset of $\mathcal{P}_\kappa\lambda$ that is homogeneous for F .

We first show that H cannot be 1-homogeneous. Suppose it were. Fix $x \in H$. Then, for any $y_0, y_1 \in H$ with $x \subsetneq y_0 \subsetneq y_1$, $d_{y_0} \cap x \neq d_{y_1} \cap x$. Let $\bar{\kappa}$ be the cardinality of $\mathcal{P}(x)$. Then $\bar{\kappa} < \kappa$, since κ is inaccessible. But there is a sequence $\langle y_\alpha \mid \alpha < \bar{\kappa}^+ \rangle$ with $x \subsetneq y_0$ and $y_\alpha \subsetneq y_\beta$ for all $\alpha < \beta < \bar{\kappa}^+$, because $\bar{\kappa}^+ < \kappa$ and κ is regular. This is a contradiction, because for all such α, β , we would have that $d_{y_\alpha} \cap x \neq d_{y_\beta} \cap x$, giving us $\bar{\kappa}^+$ distinct subsets of x .

Thus, H is 0-homogeneous. Set

$$B = \bigcup \{d_y \cap x \mid x, y \in H \text{ and } x \subsetneq y\}.$$

It follows that:

$$(1) \quad \text{If } x \subsetneq y \text{ with } x, y \in H \text{ then } B \cap x = d_y \cap x.$$

Proof of (1). The inclusion from right to left is clear, by definition of B . For the converse suppose $\alpha \in B \cap x$. Let $x' \subsetneq y'$, $x', y' \in H$, with $\alpha \in d_{y'} \cap x'$. Pick $z \in H$ with $y \cup y' \subsetneq z$. Then $\alpha \in d_{y'} \cap x' = d_z \cap x'$, so $\alpha \in d_z \cap x$, since $\alpha \in x$. So $\alpha \in d_z \cap x = d_y \cap x$, as claimed. $\square_{(1)}$

It follows that B is a strong branch, as verified by H . To see this, let $x \in \mathcal{P}_\kappa \lambda$ be given. Find $x' \in H$ with $x \subseteq x'$, and let $x \subsetneq t$, $t \in H$. We claim that for every $u \in H$ with $t \subseteq u$, $B \cap x = d_u \cap x$. But this is immediate, since $x' \subsetneq u$ and $x', u \in H$, so by (1), $B \cap x' = d_u \cap x'$, which implies that $B \cap x = d_u \cap x$, since $x \subseteq x'$. \square

One approach to generalizing the theory of ideals on κ to $\mathcal{P}_\kappa \lambda$ involved working with the ordering

$$x < y \iff x \subseteq y \wedge |x| < |\kappa \cap y|$$

instead of set inclusion - this was spearheaded by Donna Carr (see [Kan03] for the history). This leads to a natural weakening of the partition properties, namely, for $n \in \omega$ and a subset $X \subseteq \mathcal{P}_\kappa \lambda$, write $[X]_\kappa^n$ for the set $\{\{a_0, a_1, \dots, a_{n-1}\} \mid a_0 < a_1 < \dots < a_{n-1} \in X\}$, and say that $\text{Part}(\kappa, \lambda)_\kappa^n$ holds if for every function $F : [\mathcal{P}_\kappa \lambda]_\kappa^n \rightarrow 2$, there is an unbounded set $H \subseteq \mathcal{P}_\kappa \lambda$ that's homogeneous for F , meaning that $F \upharpoonright [H]^n$ is constant. It is easy to see that the proof of the previous theorem actually shows the following corollary; one just has to replace every instance of " \subsetneq " in the proof with " $<$ ".

Corollary 3.26. $\text{Part}(\kappa, \lambda)_\kappa^3$ implies that every $\mathcal{P}_\kappa \lambda$ -list has a strong branch (i.e., that κ is mildly λ -ineffable, or $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{unbounded})$ fails).

Finally, towards characterizing mild ineffability and strong compactness, we need a slight variation of the split principle.

Definition 3.27. Let \mathcal{F} be a set of functions from $\mathcal{P}_\kappa \lambda$ to $\mathcal{P}_\kappa \lambda$, and let \mathcal{B} be a family of subsets of $\mathcal{P}_\kappa \lambda$. Then $\text{Split}_{\mathcal{P}_\kappa \lambda}(\mathcal{F}, \mathcal{B})$ is the principle saying that there is a $\mathcal{P}_\kappa \lambda$ -list \vec{d} such that for every function $f \in \mathcal{F}$, there is a $\beta < \lambda$ such that both $f_\beta^+ = \{x \mid \beta \in x \wedge \beta \in d_{f(x)}\}$ and $f_\beta^- = \{x \mid \beta \in x \wedge \beta \notin d_{f(x)}\}$ belong to \mathcal{B} .

Let delay-functions be the set of delay functions from $\mathcal{P}_\kappa \lambda$ to $\mathcal{P}_\kappa \lambda$, i.e., the set of functions $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$ such that for all $x \in \mathcal{P}_\kappa \lambda$, $x \subseteq f(x)$.

Theorem 3.28. Let κ be regular and $\lambda \geq \kappa$ be a cardinal. Then the following are equivalent:

- (1) κ is mildly λ -ineffable.
- (2) $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \text{nonempty})$ fails.
- (3) $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \text{covering})$ fails.
- (4) $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \text{unbounded})$ fails.
- (5) $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \text{stationary})$ fails.

So κ is mildly ineffable iff $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \mathcal{B})$ fails, for some (equivalently, all) \mathcal{B} with $\text{stationary} \subseteq \mathcal{B} \subseteq \text{nonempty}$. It follows that κ is strongly compact iff these equivalent conditions hold for every λ .

Proof. 1. \implies 2.: Suppose $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \text{nonempty})$ held. Let \vec{d} witness this. Since κ is mildly λ -ineffable, \vec{d} has a cofinal branch $B \subseteq \lambda$. There is then a delay function $f : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$ such that for every $x \in \mathcal{P}_\kappa \lambda$, $B \cap x = d_{f(x)} \cap x$. By $\text{Split}_{\mathcal{P}_\kappa \lambda}(\text{delay-functions}, \text{nonempty})$, let β be such that both f_β^+ and f_β^- are nonempty. Let x_0, x_1 be such that $x_0 \in f_\beta^+$ and $x_1 \in f_\beta^-$. This means that $\beta \in d_{f(x_0)}$ and $\beta \notin d_{f(x_1)}$. Note that by definition, $\beta \in x_0, x_1$. But $d_{f(x_0)} = B \cap x_0$, so $\beta \in B$, while on the other hand, $d_{f(x_1)} = B \cap x_1$, so $\beta \notin B$, a contradiction.

2. \implies 3. \implies 4. \implies 5. is trivial.

5. \implies 1.: To prove that κ is mildly λ -ineffable, let \vec{d} be a $\mathcal{P}_\kappa\lambda$ -list. We have to find a cofinal branch. Since $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{delay-functions}, \text{stationary})$ fails, \vec{d} is not a $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{delay-functions}, \text{stationary})$ sequence. This means that there is a delay function f on $\mathcal{P}_\kappa\lambda$ that is not split into stationary sets by any $\beta < \lambda$ with respect to \vec{d} . As in previous arguments, this means that exactly one of f_β^+ and f_β^- is stationary (note that $\widehat{\{\beta\}}$ is the disjoint union of $f_\beta^+ \cup f_\beta^-$). So for every $\beta < \lambda$, there is a club set C_β in $\mathcal{P}_\kappa\lambda$ that's disjoint from the nonstationary one of f_β^+ and f_β^- . Let

$$B = \{\beta < \lambda \mid f_\beta^+ \text{ is stationary}\}$$

Let $D = \Delta_{\beta < \lambda} C_\beta$. To see that B is a cofinal branch, let $x \in \mathcal{P}_\kappa\lambda$ be given. Let $x \subseteq x' \in D$. We claim that $B \cap x = d_{f(x')} \cap x$. This completes the proof, since $x \subseteq x' \subseteq f(x')$. So let $\beta \in x$. Then $\beta \in x'$, and so, $x' \in C_\beta$. If $\beta \in B$, then f_β^+ is stationary, so $C_\beta \cap f_\beta^- = \emptyset$. But again, since $\beta \in x'$, it follows that $x' \in f_\beta^+$, so that $\beta \in d_{f(x')}$. If $\beta \notin B$, then $f_\beta^+ \cap C_\beta = \emptyset$, and since $\beta \in x'$, it follows that $x' \in f_\beta^-$, so $\beta \notin d_{f(x')}$.

The point about the failure of $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{delay-functions}, \mathcal{B})$ follows now, because 2. implies the failure of $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{delay-functions}, \mathcal{B})$, and this implies 5.

The claim about κ being strongly compact now follows because it is easy to see that κ is mildly λ -ineffable iff every binary (κ, λ) -mess is solvable (see [Car81, Thm. 1.4, p. 35]), and Jech showed in [Jec73, 2.2, p. 174] that every binary (κ, λ) -mess is solvable iff κ is λ -compact. \square

3.2. Split characterizations of two cardinal versions of Shelah cardinals. We will now introduce versions of the split principle whose failure can capture variants of the notion of κ being λ -Shelah. This large cardinal notion was introduced by Carr, (see [Car81]).

Definition 3.29. Let $A \subseteq \mathcal{P}_\kappa\lambda$, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)$. Then we write $\mathcal{A} \upharpoonright A := \mathcal{A} \cap \mathcal{P}(A)$.

A *functional A-list* is a sequence $\vec{f} = \langle f_x \mid x \in A \rangle$ of functions $f_x : x \longrightarrow x$. A functional A -list *splits* a set X into $\mathcal{B} \upharpoonright A$ if there is a pair $\langle \beta, \delta \rangle \in \lambda \times \lambda$ such that the sets

$$X_{\beta, \delta}^+ = \{x \in X \cap A \mid \beta \in x \wedge f_x(\beta) = \delta\} \text{ and } X_{\beta, \delta}^- = \{x \in X \cap A \mid \beta, \delta \in x \wedge f_x(\beta) \neq \delta\}$$

are in $\mathcal{B} \upharpoonright A$. The principle $A\text{-Split}_{\mathcal{P}_\kappa\lambda}^f(\mathcal{A}, \mathcal{B})$ says that there is an A -list that splits every $X \in \mathcal{A} \upharpoonright A$ into $\mathcal{B} \upharpoonright A$.

Given a functional A -list \vec{f} , a function $F : \lambda \longrightarrow \lambda$ is a

- *cofinal branch* for \vec{f} if for every $x \in \mathcal{P}_\kappa\lambda$, there is a $y \in \mathcal{P}_\kappa\lambda$ with $x \subseteq y$, such that $f_y \upharpoonright x = F \upharpoonright x$.
- *strong branch* for \vec{f} if there is an unbounded set $U \subseteq A$ such that for every $x \in \mathcal{P}_\kappa\lambda$, there is a $y \in \mathcal{P}_\kappa\lambda$ with $x \subseteq y$, such that for every $z \in U$ with $y \subseteq z$, $f_z \upharpoonright x = F \upharpoonright x$, and in this case, we say that F is *guided* by U .
- *almost ineffable branch* for \vec{f} if for unboundedly many x , $f_x = F \upharpoonright x$.
- *ineffable branch* for \vec{f} if for stationarily many x , $f_x = F \upharpoonright x$.

A is λ -*Shelah* if every functional A -list has a cofinal branch, and κ is called λ -Shelah if $\mathcal{P}_\kappa\lambda$ is λ -Shelah. A is *wildly* λ -Shelah if every functional A -list has a strong branch, and κ is *wildly* λ -Shelah if $\mathcal{P}_\kappa\lambda$ is wildly λ -Shelah.

Note that $f_x(\beta) = \delta$ can be equivalently expressed by saying that $\langle \beta, \delta \rangle \in f_x$, so the concept of a splitting functional list is a direct generalization of a splitting list. It was shown in [Car81] that κ is (almost) λ -ineffable iff every functional $\mathcal{P}_\kappa\lambda$ -list has an (almost) ineffable branch. So moving from $\mathcal{P}_\kappa\lambda$ -lists to functional $\mathcal{P}_\kappa\lambda$ -lists does not make a difference for these concepts. Obviously,

the logical relationship between these various types of branches for functional $\mathcal{P}_\kappa\lambda$ -lists is as for regular $\mathcal{P}_\kappa\lambda$ -lists, see Observation 3.11. As a consequence, if κ is almost λ -ineffable, then it is wildly λ -Shelah, and this implies λ -Shelahness and wild λ -ineffability. It was shown in [Car81] that the ideal corresponding to the failure of the Shelah property is normal, while the non-mildly ineffable ideal is equal to the ideal of the non-unbounded sets (assuming the corresponding large cardinal property).

Theorem 3.30. *Let $A \subseteq \mathcal{P}_\kappa\lambda$. A functional A -list \vec{f} splits all unbounded subsets of A iff it does not have a strong branch.*

Proof. We show both implications separately.

\Rightarrow : Towards a contradiction, assume F is a strong branch through the A -list \vec{f} , guided by the unbounded set $U \subseteq A$. Let $g : \mathcal{P}_\kappa\lambda \rightarrow U$ be so that for all $x \in \mathcal{P}_\kappa\lambda$, $g(x) \supseteq x$ and for all $z \supseteq g(x)$ with $z \in U$, $f_z \restriction x = F \restriction x$. Let $\langle \beta, \delta \rangle$ split $\tilde{U} = \text{range}(g)$ with respect to \vec{f} .

If $F(\beta) = \delta$, then we choose $y \in \tilde{U}_{\beta,\delta}^-$ with $y \supseteq g(\{\beta\})$. Then $f_y(\beta) = F(\beta) = \delta$, but since $y \in \tilde{U}_{\beta,\delta}^-$, $f_y(\beta) \neq \delta$, a contradiction. If $F(\beta) \neq \delta$, then we instead choose $y \in \tilde{U}_{\beta,\delta}^+$ with $y \supseteq g(\{\beta\})$ and get the same contradiction.

\Leftarrow : We will show the contrapositive. So suppose \vec{f} is a functional A -list that does not split unbounded sets. There is then an unbounded $U \subseteq \mathcal{P}_\kappa\lambda$ such that for each $\langle \beta, \delta \rangle \in \lambda \times \lambda$, exactly one of $U_{\langle \beta, \delta \rangle}^+$ and $U_{\langle \beta, \delta \rangle}^-$ is unbounded in $\mathcal{P}_\kappa\lambda$. Define $F \subseteq \lambda \times \lambda$ by setting

$$\langle \beta, \delta \rangle \in F \iff U_{\beta,\delta}^+ \text{ is unbounded in } \mathcal{P}_\kappa\lambda.$$

We claim that U guides F , making it a strong branch.

To see this, fix $x \in \mathcal{P}_\kappa\lambda$. For each pair $\langle \beta, \delta \rangle \in x \times x$, there has to be a $y_{\langle \beta, \delta \rangle} \in \mathcal{P}_\kappa\lambda$ such that either for all $z \supseteq y_{\langle \beta, \delta \rangle}$ with $z \in U$ we have that $f_z(\beta) = \delta$, or for all $z \supseteq y_{\langle \beta, \delta \rangle}$ with $z \in U$, $f_z(\beta) \neq \delta$. The former holds if $U_{\beta,\delta}^-$ is not unbounded, and the latter holds if $U_{\beta,\delta}^+$ is not unbounded. Let $y = \bigcup_{\langle \beta, \delta \rangle \in x \times x} y_{\langle \beta, \delta \rangle}$, and pick $z \supseteq y$ with $z \in U$.

Now $F \cap (x \times x) = f_z \cap (x \times x)$: From right to left, suppose $\langle \beta, \delta \rangle \in x \times x$ and $f_z(\beta) = \delta$. Then for all $z' \supseteq z$ with $z' \in U$, $f_{z'}(\beta) = \delta$, because $z' \supseteq z \supseteq y_{\langle \beta, \delta \rangle}$. So $U_{\langle \beta, \delta \rangle}^+$ is unbounded, and hence, $\langle \beta, \delta \rangle \in F$. Vice versa, if $\langle \beta, \delta \rangle \in F \cap (x \times x)$, then $U_{\beta,\delta}^+$ is unbounded, and hence, for all $z' \supseteq y_{\langle \beta, \delta \rangle}$ with $z' \in U$, $f_{z'}(\beta) = \delta$ (because the alternative would be that for all $z' \supseteq y_{\langle \beta, \delta \rangle}$ with $z' \in U$, $f_{z'}(\beta) \neq \delta$, but that would mean that $U_{\langle \beta, \delta \rangle}^+$ is not unbounded). So since $z \supseteq y_{\langle \beta, \delta \rangle}$ and $z \in U$, $f_z(\beta) = \delta$, as claimed.

This implies that F is a function, and hence that it is a strong branch in the functional sense. \square

Lemma 3.31. *Let κ be regular. κ is wildly λ -Shelah iff $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded})$ fails. It follows that κ is supercompact iff $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded})$ fails, for all λ .*

Proof. It follows from [Car81] and [Mag74] that κ is supercompact iff κ is λ -Shelah for all λ iff κ is almost λ -ineffable for all λ iff κ is λ -ineffable for all λ . Moreover, [Car81, p. 52, Cor. 1.4] shows that if κ is almost λ -ineffable, then every functional $\mathcal{P}_\kappa\lambda$ -list has an almost ineffable branch, and if κ is λ -ineffable, then every functional $\mathcal{P}_\kappa\lambda$ -list has an ineffable branch. In particular, if κ is almost λ -ineffable, then it is wildly λ -Shelah. It follows from all of this that the failure of $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded})$ for every λ characterizes the supercompactness of κ . \square

In order to characterize when κ is λ -Shelah, we need a modification of the functional split principles similar to the modification that was needed in order to characterize mild ineffability.

Definition 3.32. Let \mathcal{F} be a set of functions from $\mathcal{P}_\kappa\lambda$ to $\mathcal{P}_\kappa\lambda$, and let \mathcal{B} be a family of subsets of $\mathcal{P}_\kappa\lambda$. Then $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\mathcal{F}, \mathcal{B})$ is the principle saying that there is a functional list \vec{f} such that for every function $g \in \mathcal{F}$, there is a pair $\langle \beta, \delta \rangle \in \lambda \times \lambda$ such that both $g_{\langle \beta, \delta \rangle}^+ = \{x \mid \beta \in x \wedge f_{g(x)}(\beta) = \delta\}$ and $g_{\langle \beta, \delta \rangle}^- = \{x \mid \beta \in x \wedge f_{g(x)}(\beta) \neq \delta\}$ belong to \mathcal{B} .

Lemma 3.33. Let κ be regular and $\lambda \geq \kappa$ be a cardinal. The following are equivalent:

- (1) κ is λ -Shelah
- (2) $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{delay-functions, nonempty})$ fails.
- (3) $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{delay-functions, covering})$ fails.
- (4) $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{delay-functions, unbounded})$ fails.
- (5) $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{delay-functions, stationary})$ fails.

It follows that κ is supercompact iff these conditions hold for arbitrarily large λ .

Proof. The proof of 3.28 goes through with minor modifications. By [Car81, p. 63, Cor. 2.1], κ is supercompact iff κ is λ -Shelah, for every λ . \square

4. SPLIT IDEALS

In the study of $\mathcal{P}_\kappa\lambda$ -combinatorics, it has proven fruitful to investigate ideals associated to various large cardinal properties. This was done, for example, for the ideal $\text{NIn}_{\kappa,\lambda}$ of non-ineffable subsets of $\mathcal{P}_\kappa\lambda$, the ideal $\text{NAln}_{\kappa,\lambda}$ of non-almost-ineffable subsets and the ideal $\text{NMl}_{\kappa,\lambda}$ of non-mildly-ineffable subsets of $\mathcal{P}_\kappa\lambda$ in [Car81]. It was shown there that if κ is mildly λ -ineffable, then $\text{NMl}_{\kappa,\lambda}$ is equal to the ideal $\text{I}_{\kappa,\lambda}$ of non-unbounded subsets of $\mathcal{P}_\kappa\lambda$, and that the other ideals are normal ideals if κ has the corresponding large cardinal property. Since the split principles characterize the failure of a large cardinal property, they allow us to define such ideals in a natural way. Since some of the large cardinal properties sprouting from our investigation of split principles, such as wild ineffability, appear to be new, it seems worthwhile to investigate these ideals.

Definition 4.1. Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathcal{P}_\kappa\lambda)$ be families of subsets of $\mathcal{P}_\kappa\lambda$, and let $A \subseteq \mathcal{P}_\kappa\lambda$. Then we write $\mathcal{A} \upharpoonright A$ for $\mathcal{A} \cap \mathcal{P}(A)$, i.e., for the family of sets in \mathcal{A} that are contained in A . We let

$$\mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A}, \mathcal{B})) = \{A \subseteq \mathcal{P}_\kappa\lambda \mid \text{Split}_\kappa(\mathcal{A} \upharpoonright A, \mathcal{B}) \text{ holds}\}$$

In the more natural cases, $\mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\mathcal{A}, \mathcal{B}))$ is an ideal - this is the case if $\mathcal{A} = I^+$, for an ideal I on $\mathcal{P}_\kappa\lambda$. The proof of Observation 2.17 goes through.

We will first work with the principle $\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded})$, whose failure characterizes the wildly ineffable cardinals.

Lemma 4.2. $\mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}))$ is a κ -complete ideal containing $\text{I}_{\kappa,\lambda}$.

Proof. Clearly, $\text{I}_{\kappa,\lambda} \subseteq \mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}))$, since a non-unbounded set has no unbounded subsets. To see that it is an ideal, first observe that if $Y \in \mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}))$ and $X \subseteq Y$, then $X \in \mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}))$, because if \vec{d} is a Y -list that splits every unbounded subset of Y , then $\vec{d} \upharpoonright X$ is an X -list that splits every unbounded subset of X . Second, we show κ -completeness. Thus, let $\bar{\kappa} < \kappa$ and let $\langle X_\gamma \mid \gamma < \bar{\kappa} \rangle$ be a sequence of sets in $\mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}(\text{unbounded}))$. For each $\gamma < \bar{\kappa}$, let $\langle d_x^\gamma \mid x \in X_\gamma \rangle$ be a sequence which splits every unbounded subset of X_γ . Let $X = \bigcup_{\gamma < \bar{\kappa}} X_\gamma$. We have to show that there is an X -list that splits every unbounded subset of X , and to achieve this, we amalgamate the lists \vec{d}^γ by letting, for $x \in X$, $\nu(x)$ be least such that $x \in X_{\nu(x)}$, and by setting

$$d_x = d_x^{\nu(x)}$$

for $x \in X$. We claim that \vec{d} splits every unbounded subset of X . To see this, let $Y \subseteq X$ be unbounded. Then for some $\gamma < \bar{\kappa}$, the set $\bar{Y} = \{x \in Y \mid \nu(x) = \gamma\}$ is unbounded, because the ideal of non-unbounded sets is κ -complete. Since \vec{d}^γ splits every unbounded subset of X_γ , and since \bar{Y} is an unbounded subset of X_γ , there is a β that splits \bar{Y} with respect to \vec{d}^γ . But $\vec{d}^\gamma \upharpoonright \bar{Y} = \vec{d} \upharpoonright \bar{Y}$, so β splits \bar{Y} with respect to \vec{d} . \square

Our lack of knowledge about the relationship between mild ineffability and wild ineffability is reflected by some open questions about the split ideal. It was shown in [Car81] that if κ is mildly λ -ineffable, then $\text{NMI}_{\kappa,\lambda} = \mathbf{I}_{\kappa,\lambda}$. We do not know whether this is true of the split ideal, and we do not know whether the split ideal is normal, assuming that κ is wildly λ -ineffable.

The ideal corresponding to wild λ -Shelahness, on the other hand, is normal, like the one corresponding to λ -Shelahness. The latter was shown by Carr, and her proof generalizes very directly.

Recall that an ideal I on $\mathcal{P}_\kappa\lambda$ is *normal* if for every sequence $\langle X_\nu \mid \nu < \lambda \rangle$ of members of I , the diagonal union

$$\nabla_{\nu < \lambda} X_\nu = \{x \in \mathcal{P}_\kappa\lambda \mid \exists \nu \in x \quad x \in X_\nu\}$$

belongs to I .

Theorem 4.3. *κ is wildly λ -Shelah iff $I := \mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded}))$ is a normal proper ideal on $\mathcal{P}_\kappa\lambda$.*

Proof. The direction from right to left is trivial, since if I is a proper ideal, then $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded})$ fails, which implies that κ is wildly λ -Shelah, by Theorem 3.30.

For the substantial forward direction, assume that κ is wildly Shelah. According to [Car81, Lemma 2.2], to show that I is a normal proper ideal on $\mathcal{P}_\kappa\lambda$ it suffices to show that (0) I is a proper ideal, (1) I is closed under subsets, (2) if $X \in I$ and $Y \in \mathbf{I}_{\kappa,\lambda}$, then $X \cup Y \in I$ and (3) I is closed under diagonal unions.

(0) is clear by our assumption that κ is wildly λ -Shelah. (1) is obvious, as in Lemma 4.2. (2) is clear because if $X \in I$ and Y is not unbounded, then we can let \vec{f} be a functional X -list splitting every unbounded subset of X , and extend it arbitrarily to a functional $X \cup Y$ -list \vec{f}' . If $A \subseteq X \cup Y$ is unbounded, then $A = (A \cap X) \cup (A \cap Y)$, so one of $A \cap X$ and $A \cap Y$ is unbounded, as the non-unbounded sets form an ideal. Clearly then, $A \cap X$ is unbounded, so split by \vec{f} , and hence, A is split by \vec{f}' .

The crucial point is (3), the closure of I under diagonal unions. So let $\langle X_\nu \mid \nu < \lambda \rangle$ be a sequence with $X_\nu \in I$ for all $\nu < \lambda$. Fix, for every such ν , a functional X_ν -list $\langle f_x^\nu \mid x \in X_\nu \rangle$ that splits every unbounded subset of X_ν , and let $X' = \nabla_{\nu < \lambda} X_\nu$. That is, for $x \in \mathcal{P}_\kappa\lambda$, $x \in X'$ iff there is $\nu \in x$ such that $x \in X_\nu$. For $x \in X'$, let $\gamma(x) \in x$ such that $x \in X_{\gamma(x)}$. We follow the proof of [Car81, Thm. 2.3] closely here.

Let $\widehat{\{0\}} = \{x \in \mathcal{P}_\kappa\lambda \mid 0 \in x\}$. Let $X = X' \cap \widehat{\{0\}} = \{x \in X' \mid 0 \in x\}$. It suffices to show that $X \in I$, since then it follows by (2) that $X' = X \cup (X' \setminus \widehat{\{0\}}) \in I$, as $X' \setminus \widehat{\{0\}} \in \mathbf{I}_{\kappa,\lambda}$.

For every $x \in \mathcal{P}_\kappa\lambda$, let $\langle \alpha_\xi^x \mid \xi < \text{otp}(x) \rangle$ be the monotone enumeration of x . Since for every $x \in X$ we have $0 \in x$, it follows that $\alpha_0^x = 0$. We amalgamate the functional lists \vec{f}^ν into one functional X -list g by defining $g_x : x \rightarrow x$, for $x \in X$, as follows.

$$g_x(\alpha_\xi^x) = \begin{cases} \gamma(x) & \text{if } \xi = 0 \text{ or } \xi \text{ is a limit ordinal,} \\ f_x^{\gamma(x)}(\alpha_{\xi-1}^x) & \text{if } \xi \text{ is a successor ordinal} \end{cases}$$

for $\xi < \text{otp}(x)$. Assuming that X is not in I , the functional split ideal on $\mathcal{P}_\kappa\lambda$, no functional X -list splits all unbounded subsets of X , so Theorem 3.30 implies that every functional X -list has a strong branch. Let $G : \lambda \rightarrow \lambda$ be a strong branch for \vec{g} , guided by the unbounded set $U \subseteq X$. Let $\gamma = G(0)$, and define $F : \lambda \rightarrow \lambda$ by $F(\xi) = G(\xi + 1)$.

We claim that F is a strong branch for \vec{f}^γ , guided by $U \cap X_\gamma$.

To see this, let $x \in \mathcal{P}_\kappa\lambda$. Set $x' = x \cup \{0\} \cup \{\xi + 1 \mid \xi \in x\}$. Let $y \in \mathcal{P}_\kappa\lambda$ with $x' \subseteq y$ be such that for all $z \in U$ with $y \subseteq z$, $G \restriction x' = g_z \restriction x'$. Since $0 \in x'$, it follows that $g_z(0) = G(0) = \gamma$. So for every $\xi \in x$, we have that $\xi, \xi + 1 \in x'$, so $F(\xi) = G(\xi + 1) = g_z(\xi + 1) = f_z^\gamma(\xi)$. Note that since U is unbounded, there are such z (meaning $z \in U$ with $y \subseteq z$), and for every such z , since $g_z(0) = G(0) = \gamma = \gamma(z)$, it follows that $z \in X_\gamma$. So $U \cap X_\gamma$ is unbounded.

We have reached a contradiction, since we assumed that \vec{f}^γ splits all unbounded subsets of X_γ , which implies, by Theorem 3.30, that it does not have a strong branch. \square

Definition 4.4. Let I be an ideal on $\mathcal{P}_\kappa\lambda$.

I is *strongly normal* iff every function $f : X \rightarrow \mathcal{P}_\kappa\lambda$ such that $X \in I^+$ and for every $x \in X$, $f(x) < x$, it follows that there is a y such that $f^{-1}[\{y\}] \in I^+$.

Using methods from [Car87], it is not hard to improve the previous theorem as follows, assuming $\lambda^{<\kappa} = \lambda$.

Theorem 4.5. Suppose κ is wildly λ -Shelah, where $\lambda^{<\kappa} = \lambda$. Then $I := \mathcal{I}(\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded}))$ is a strongly normal ideal on $\mathcal{P}_\kappa\lambda$.

Note: It was shown in [Joh90] that if κ is λ -Shelah and $\text{cf}(\lambda) \geq \kappa$, then $\lambda^{<\kappa} = \kappa$.

Proof. First, let's write $\text{NSh}_{\kappa,\lambda}$ for the ideal of subsets X of $\mathcal{P}_\kappa\lambda$ that are not λ -Shelah. Clearly then, $\text{NSh}_{\kappa,\lambda} \subseteq I$, since if $X \subseteq \mathcal{P}_\kappa\lambda$ is not λ -Shelah, then $\text{Split}_{\mathcal{P}_\kappa\lambda}^f(\text{unbounded} \restriction X)$ holds, or else, X would be wildly λ -Shelah, and hence λ -Shelah. As a result, the same relation holds between the dual filters associated with these ideals: $\text{NSh}_{\kappa,\lambda}^* \subseteq I^*$.

Fix a bijection $\varphi : \mathcal{P}_\kappa\lambda \rightarrow \lambda$. It was shown in [Car87, Prop. 3.4] that if κ is λ -Shelah and $\lambda = \lambda^{<\kappa}$ it follows that the sets $A = \{x \in \mathcal{P}_\kappa\lambda \mid x \cap \kappa \text{ is an inaccessible cardinal}\}$ and $B = \{x \in \mathcal{P}_\kappa\lambda \mid \varphi[\mathcal{P}_{\kappa_x}(x)] = x\}$ (where $\kappa_x = |x \cap \kappa|$) belong to $\text{NSh}_{\kappa,\lambda}^*$. It follows that they belong to I^* .

Using these facts, the proof of [Car87, Thm. 3.5] goes through, to show the claim. Namely, given $X \in I^+$ (i.e., a set $X \subseteq \mathcal{P}_\kappa\lambda$ that is wildly λ -ineffable) and a function $f : X \rightarrow \mathcal{P}_\kappa\lambda$ such that for all $x \in X$, $f(x) < x$, we have to show that there is a $y \in \mathcal{P}_\kappa\lambda$ such that $f^{-1}[\{y\}] \in I^+$. Let $X_1 = X \cap B$, and note that $X_1 \in I^+$. Define $g : X_1 \rightarrow \lambda$ by $g(x) = \varphi(f(x))$. Then $g(x) \in x$, and hence g is regressive. Since I is normal, by Theorem 4.3, there is an $\alpha < \lambda$ such that $g^{-1}[\{\alpha\}]$ is in I^+ . But $g^{-1}[\{\alpha\}] = f^{-1}[\{\varphi^{-1}(\alpha)\}]$, so we are done. \square

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